A STUDY OF CONCAVE AND CONVEX PARABOLIC FINS AND SPINES

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By
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CERTIFICATE

This is to certify that the work on "A study of concave and convex parabolic longitudinal fins and spines", by Karra Sreenivas, has been carried out under my supervision and has not been submitted elsewhere for a degree.

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I offer my profound gratitude to Dr. P.N. Kaul under whose able guidance the present work has been carried out.

Rather than risk omission of a single name, I hope that a grateful general acknowledgement will express my sincere thanks to all my friends who have made my stay here a memorable one.

KARRA SREENIVAS

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NOMENCLATURE

A	CROSS SECTIONAL AREA OF THE FIN NORMAL TO x (m^2)
A_s	Surface area of the fin (m^2)
Bi	Biot number at the base of the fin ($h_f L/k$) non-dimensiona
h_a	Heat Transfer coefficient between the fin and the surrounding fluid ($W/m^2.K$)
h_f	Heat Transfer coefficient between the fin base and the fluid past the base wall ($W/m^2.K$)
$I_n(x)$	Modified Bessel function of the first kind and order n
k	Thermal conductivity of the fin material ($W/m.K$)
L	Length of the fin (m)
N	A fin parameter ($\left(\frac{2h_a L^2}{kw}\right)^{1/2}$)
R	Radius of the spine (m)
T	Temperature (K)
T_a	Ambient temperature (K)
T_f	Temperature of the fluid past the fin base wall (K)
w	The width of the fin (m)
X	Dimensionless parameter (x/L)
$K_n(x)$	Modified bessel function of the second kind and order n.

δ	Thickness of the fin base wall
δ^*	A dimensionless parameter, corresponding to the thickness ratio (δ/L)
Θ	Dimensionless parameter (T/T_a)
Θ_f	Dimensionless parameter (T_f/T_a)
Ψ	A dimensionless parameter defined as (Θ -1)

SYNOPSIS

Theoretical analyses for finding the optimal dimensions of longitudinal fins and spines of concave and convex parabolic profiles have been carried out. In the present analysis it has been assumed that convective boundary conditions exist at the fin base wall. The resistance offered to heat flow by the fin base wall has been accounted for.

The analyses derive the heat transfer rate for each type of the fin. The optimization procedure then leads to a relationship between the parameters $N=\left(\frac{2h_a L^2}{kw}\right)^{1/2}$, $\delta^*=\delta/L$ and $B_i=\frac{h_f L}{k}$, which must be satisfied to obtain the optimum dimensions for that particular fin. The relationships are presented in the form of graphs. Software has been developed for design applications. Program listing is included in the thesis. Concave parabolic spine is found to be the most efficient of the four fin types studied in this work.

CHAPTER-I

INTRODUCTION

1.1 INTRODUCTION :

In the design and construction of various types of heat transfer equipment, an extended surface is attached to the prime surface specifically to enhance the heat transfer rate between a solid and an adjoining fluid. Such an extended surface is termed a fin. The heat is conducted through the material and finally dissipated to the surroundings by convection. An analysis of the combined conduction-convection systems is therefore important from the practical stand point.

Fins offer a trouble free and compact method of enhancing heat transfer. Extended surface applications are numerous. Examples may be found in chemical, refrigeration and cryogenic processes ; electrical and electronic components, nuclear fuel modules, cooling fins of air cooled engines, the fin extensions to the tubes of radiators and other heat exchangers.

A straight fin is any extended surface that is attached to a plane wall. It may be of uniform cross sectional area or it's cross sectional area may vary with the distance x from the base wall. A pin fin or spine is an extended surface of circular cross section. Spines can be of uniform or non uniform cross section.

Fins of different geometries and materials respond differently to identical and uniform heat sources or sinks. Important to the analyses of fin geometries are the constraints or the assumptions which are usually employed to define and limit the problem, with a view to simplifying it's solution. These assumptions for conventional analysis are :

(1) Steady state conduction takes place in the fin

(2) Fin material is isotropic

(3) Heat transfer coefficient is constant over the entire surface of the fin

(4) Temparature of the fin surroundings is uniform

(5) Fin thickness is so small compared with it's length that the temperature gradients across fin thickness may be neglected

(6) Temperature at the base of the fin is known and uniform

(7) Resistance offered to the flow of heat by the fin base wall is small and may be neglected

(8) Heat flow from the fin tip is negligible

(9) Contact resistance between the fin and prime surface is negligible

(10) There are no heat sources or sinks within the fin

(11) Heat transfer to or from the fin is proportional to the temperature excess between the fin and the surrounding medium.

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For the fin to dissipate a maximum amount of heat for a minimum value of fin material used, it is necessary to use proper fins in modern designs. An optimally designed fin would thus minimize the cost, the space requirement and the weight of the system.

One can optimize the fins in one of the following two ways. One is to design the shape of the fin for a maximum heat dissipation, when dimensions of the fin are fixed. The other way is to find the dimensions of the fin for maximum heat transfer, after the shape has been decided upon.

1.2 LITERATURE REVIEW :

As mentioned before, the traditional approach to the optimization of extended surfaces has been, to minimize the consumption of fin material for the maximisation of heat transmission.

Earlier optimization procedures carried out by Shmidt (1) , Duffin (2) , Brown (3) , Murray (4) and Gardner (5) were based on the conventional fin analysis assumptions mentioned in the introduction. Kern and Kraus (6) have summarised many of these contributions in their book.

The suggestion that a realistic optimization analysis must employ a convective boundary condition at the base of the fin wall and include the conduction resistance of the fin base wall was made by Suryanarayana (7) in 1977. This suggestion

amounts to the relaxation of the assumptions 6 and 7 of the conventional fin design analysis. Using 2-D finite difference method he demonstrated that the error introduced into the results due to these two assumptions was significant.

An optimization analysis based on a convective boundary condition was carried out by Aziz (8) in 1985. But he did not take into account the resistance offered to the flow of heat by the fin base wall.

An optimization analysis for various longitudinal and radial fins, by Relaxing the assumptions 6, 7 and 8 in the conventional fin analysis was carried out by Alex (10) in 1986.

1.3 THE PRESENT WORK.

The optimization analyses carried out so far, as mentioned earlier, were based on a number of assumptions. Though Alex (10) carried out his analysis relaxing some of those assumptions he left out some of the fin profiles. Not much work has been done to relax these assumptions at least for the fin profiles considered in this work.

It is not realistic to assume a known base temperature, since in an actual situation of heat dissipation by a fin the heat transfer is governed by convection. For a more accurate analysis one must apply a convective boundary condition on the plane side of the fin base wall.

The conventional analysis of the fins also ignores the resistance offered by the fin base wall. This is also not realistic. So in this analysis we have taken into account the resistance offered by the fin base wall to the flow of heat. From the graphs for the optimum dimensions of fins (Fig.Nos.5,7,9,11), we see that the inclusion of the conduction resistance offered by the fin base wall, in the analysis, lowers the value of Bi^{-1} and N for constant values of N and B_i^{-1} respectively.

This thesis includes the results of the optimization analyses of longitudinal fins of convex and concave parabolic profiles and spines of concave and convex parabolic profiles, wi the assumption numbers 6 and 7 of the conventional analysis relaxed.

For each type of the fin the basic differential equations governing the heat transfer in the fin and fin base wall are solved simultaneously, to yield an expression for temperature excess. Next the correct temperature distribution is obtained using the appropriate boundary conditions. The equation for the heat dissipation by the fin is then obtained from the Fourier's rate equation.

The optimization procedure employed is to express the dissipation, q, as a function of, w, the width of the fin in case of longitudinal fins of concave and convex parabolic

profiles, and then to apply the condition $\frac{dq}{dw} = 0$, for maximizing the heat transfer. In case of spines of concave and convex, parabolic profiles, q, is expressed as a function of R, the radius of the spine and then the condition $\frac{dq}{dR} = 0$ is applied for maximizing the heat transfer.

For each type of fin or spine, the above optimization procedure leads to a relationship between the fin parameter, N, and the biot number, Bi , and the thickness ratio, δ^*, that must be satisfied for optimum conditions. The variation of, Bi, with , N, under different conditions are then plotted.

Since we have not assumed a known base temperature an iterative solution is required for the convective boundary condition.

For a desired heat dissipation rate, the fin design procedure would be as follows : Select a trial value of, L, the length of the fin, which together with other data fixes, Bi, and, δ^* . Using the Relationship between N, Bi and δ^*, N value can be found out. Now we calculate the heat dissipation, q, using the appropriate expression for the desired fin type and compare it with the desired heat dissipation. We repeat the procedure by incrementing the length till the values match. From the value of N, at which desired, q, is achieved we find ou the radius or width of the fin under consideration.

CHAPTER- II

MATHEMATICAL FORMULATION AND OPTIMIZATION ANALYSIS

2.1 LONGITUDINAL FIN OF CONCAVE PARABOLIC PROFILE.

Derivation of the governing differential equation for the flow of heat through the fin :

The general differential equation for any longitudinal fin is

$$2f_2(x)\,\frac{d^2T}{dx^2} + 2\,\frac{df_2(x)}{dx}\,\frac{dT}{dx} - \frac{2h_a}{k}\,(T-T_a) = 0 \qquad \text{.... (2.1.1)}$$

The profile function $f_2(x)$ in case of the concave parabolic fin shown in Fig. 1, is

$$f_2(x) = \frac{w}{2}\left(\frac{x}{L}\right)^2 \qquad \text{.... (2.1.2)}$$

$$\frac{df_2(x)}{dx} = \frac{wx}{L^2} \qquad \text{.... (2.1.3)}$$

Substituting the Eqs. (2.1.2) and (2.1.3) in Eq. (2.1.1), we get

$$x^2\,\frac{d^2T}{dx^2} + 2x\frac{dT}{dx} - \frac{2h_aL^2}{kw}\,(T-T_a) = 0 \qquad \text{.... (2.1.4)}$$

Eq. (2.1.4) is then the governing differential Equation for the concave parabolic fin.

The differential equation governing the flow of heat through the fin base wall is

$$d^2T/dx^2 = 0 \qquad \ldots\ldots (2.1.5)$$

BOUNDARY CONDITIONS :

1. At the tip of the fin (x=0), $\frac{dT}{dx} = 0$

2. At the wall surface where $x = \delta+L$,

$$K\frac{dT}{dx} = h_f\ (T_f - T)$$

3. At the base of the fin i.e. at $x = L$,

$$T_1 = T_2 \text{ and } \frac{dT_1}{dx} = \frac{dT_2}{dx}$$

Nondimensionalizing the governing equations for the fin, the base wall, and the boundary conditions by using the following expressions :

$$\theta = T/T_a,\ X = \frac{x}{L},\ \theta_f = \frac{T_f}{T},\ N^2 = \frac{2h_aL^2}{kw}$$

and $\frac{dT}{dx} = \frac{T_a}{L}\frac{d\theta}{dX}, \frac{d^2T}{dx^2} = \frac{T_a}{L^2}\frac{d^2\theta}{dX^2}$

(1) For the fin, we get

$$X^2\frac{d^2\theta}{dX^2} + 2X\frac{d\theta}{dX} - N^2(\theta - 1) = 0$$

At X=0 i.e. at the tip of the fin $\frac{d\theta}{dX} = 0$

(2) For the fin base wall

$$\frac{d^2\theta}{dX^2} = 0$$

At $X = 1+\delta^*$ i.e. at the surface of the wall

$$\frac{d\theta}{dX} = B_i(\theta_f - \theta)$$

(3) At $X = 1$ i.e. at the base of the fin

$$\theta_1 = \theta_2 \text{ and } \frac{d\theta_1}{dX} = \frac{d\theta_2}{dX}$$

To solve the above equations let us assume that

$$\theta - 1 = \Psi$$

Then the equations and the boundary conditions become

(1) For the fin

$$X^2 \frac{d^2\Psi}{dX^2} + 2X \frac{d\Psi}{dX} - N^2\Psi = 0 \qquad \text{.... (2.1.6)}$$

At the tip of the fin i.e. at $X=0$, $\frac{d\Psi}{dX} = 0$ .... (2.1.7)

(2) For the fin base wall

$$\frac{d^2\Psi}{dX^2} = 0 \qquad \text{.... (2.1.8)}$$

At the surface of the wall i.e. at $X = 1+\delta^*$

$$\frac{d\Psi}{dX} = B_i(\theta_f - 1 - \Psi) \qquad \text{.... (2.1.9)}$$

(3) At the base of the fin i.e. at $X = 1$

$$\Psi_1 = \Psi_2 \qquad \text{.... (2.1.10)}$$

$$\frac{d\Psi_1}{dX} = \frac{d\Psi_2}{dX} \qquad \text{.... (2.1.11)}$$

The boundary conditions (2.1.10) and (2.1.11) are common to both the Eqs. (2.1.6) and (2.1.8). The solution of Eq. (2.1.6) is obtained as follows.

Let $X = e^{v}$ or $v = \ln X$ then

$$\frac{d\Psi}{dX} = \frac{d\Psi}{dv}\cdot\frac{dv}{dX} = \frac{1}{X}\frac{d\Psi}{dv}$$

$$\frac{d^2\Psi}{dX^2} = \frac{d\left(\frac{1}{X}\cdot\frac{d\Psi}{dv}\right)}{dx} = -\frac{1}{X^2}\cdot\frac{d\Psi}{dv} + \frac{1}{X}\frac{d}{dx}(d\Psi/dv)$$

$$= -\frac{1}{X^2}\frac{d\Psi}{dv} + \frac{1}{X^2}\frac{d^2\Psi}{dv^2}$$

Now Equation (2.1.6) can be written as

$$\frac{d^2\Psi}{dv^2} + \frac{d\Psi}{dv} - N^2\Psi = 0 \qquad \ldots. \ (2.1.12)$$

The solution of Eq. (2.1.12) is

$$\Psi = c_1\, e^{P_1 v} + c_2\, e^{P_2 v}$$

In terms of independent variable X

$$\Psi = c_1\, X^{P_1} + c_2 X^{P_2} \qquad \ldots. \ (2.1.13)$$

where $P_1 = -1/2 + 1/2 \ (1+4N^2)^{1/2}$

$P_2 = -1/2 - 1/2 \ (1+4N^2)^{1/2}$

It can be observed that at X=0, the temperature excess will be un-bounded unless $c_2=0$ in Eq. (2.1.13).

$$\therefore \Psi = c_1\, X^{P_1} \qquad \ldots. \ (2.1.14)$$

The solution of Eq. (2.1.8) is

$$\Psi = AX + B \qquad \text{.... (2.1.15)}$$

Using the boundary condition (2.1.9) in Eq. (2.1.15), we get

$$A = B_i(\theta_f - 1 - A(1+\delta^*) - B)$$

or $$B = (\theta_f - 1) - A(1/B_i + 1 + \delta^*)$$

$$\therefore \quad \Psi = AX + (\theta_f - 1) - A(1/B_i + 1 + \delta^*) \qquad \text{.... (2.1.16)}$$

Similarly using the boundary condition (2.1.10) in Eqs. (2.1.14) and (2.1.16), we obtain

$$c_1 = (\theta_f - 1) - A(1/B_i + \delta^*) \qquad \text{.... (2.1.17)}$$

Substituting the boundary condition (2.1.11) in Eqs. (2.1.14) and (2.1.16), we have

$$A = P_1 c_1 \qquad \text{.... (2.1.18)}$$

Also substituting Eq.(2.1.18) in Eq. (2.1.17) yields

$$c_1 = (\theta_f - 1) - P_1 c_1 (1/B_i + \delta^*)$$

$$\therefore \quad c_1 = \frac{(\theta_f - 1)}{1 + P_1(1/B_i + \delta^*)}$$

Substituting the value of c_1 in Eq. (2.1.14) gives

$$\Psi = (\theta_f - 1)X^{P_1}/(1 + P_1(1/B_i + \delta^*)) \qquad \text{.... (2.1.19)}$$

$$\therefore \quad \frac{d\Psi}{dX} = \frac{P_1 X^{P_1 - 1}(\theta_f - 1)}{1 + P_1(1/B_i + \delta^*)}$$

$$\text{also}\left(\frac{d\Psi}{dX}\right)_{X=1} = \frac{P_1(\theta_f-1)}{1+P_1(1/B_i+\delta^*)}$$

The heat transferred through the base of the fin is,

$$q = KA\left(\frac{dT}{dX}\right)_{X=L}$$

$$\therefore q = \frac{Kw\ T_a}{L}\cdot\frac{(\theta_f-1)P_1}{1+P_1(1/B_i+\delta^*)} \quad \text{.... (2.1.20)}$$

To optimize the heat transfer let us assume that the volume of the fin, 1/3 WL=V (a constant).

The optimization procedure is to express, q, as a function of, w, alone and then applying the condition $\frac{dq}{dw} = 0$. The application of this condition yields the following relationship between B_i, δ^* and N.

$$1/B_i = \frac{((1+4N^2)^{1/2}-N^2-1)}{(1+4N^2)^{1/2}(\ (1+4N^2)^{1/2}-1)^2} - \delta^*$$

(For derivation please refer to appendix A)

2.2 LONGITUDINAL FIN OF CONVEX PARABOLIC PROFILE :

Derivation of the governing differential equation :

The general differential equation for any longitudinal fin is

$$2f_2(x)\frac{d^2T}{dx^2} + 2\frac{df_2(x)}{dx}\cdot\frac{dT}{dx} - \frac{2h_a}{k}(T-T_a) = 0 \quad \text{.... (2.2.1)}$$

The profile function $f_2(x)$ in case of the convex parabolic fin, shown in Fig. 2, is

$$f_2(x) = \frac{w}{2} \left(\frac{x}{L}\right)^{1/2} \qquad \text{..... (2.2.2)}$$

$$\frac{df_2(x)}{dx} = \frac{w}{4\,(Lx)^{1/2}} \qquad \text{..... (2.2.3)}$$

Substituting (2.2.2) and (2.2.3) in equation (2.2.1), we get

$$x^{1/2}\frac{d^2T}{dx^2} + \frac{1}{2\,x^{1/2}}\frac{dT}{dx} - m^2\,(L)^{1/2}(T-T_a) = 0 \qquad \text{..... (2.2.4)}$$

where $m = (2\ ha/kw.)^{1/2}$

Equation (2.2.4) is the governing differential equation for the convex parabolic fin.

The differential equation governing the flow of heat through the fin base wall is,

$$\frac{d^2T}{dx^2} = 0 \qquad \text{..... (2.2.5)}$$

BOUNDARY CONDITIONS :

1. At the tip of the fin when, $x = 0, \frac{dT}{dx} = 0$
2. At the wall surface when $x = L+\delta$,

$$k\,\frac{dT}{dx} = h_f\,(T_f - T)$$

3. At the base of the fin i.e. at $x = L$,

$$T_1 = T_2 \quad \text{and} \quad \frac{dT_1}{dx} = \frac{dT_2}{dx}$$

Nondimensionalizing the governing equations for the fin, the base wall, and the boundary conditions by using the following expressions.

$$\theta = \frac{T}{T_a}, \quad X = \frac{x}{L}, \quad \theta_f = \frac{T_f}{T_a}$$

$$\frac{dT}{dx} = \frac{T_a}{L}\frac{d\theta}{dX}, \quad \frac{d^2T}{dx^2} = \frac{T_a}{L^2}\frac{d^2\theta}{dX^2}$$

Then,

(1) <u>For the fin,</u> we get

$$X^{1/2}\frac{d^2\theta}{dX^2} + \frac{X^{-1/2}}{2} \quad \frac{1}{2X^{1/2}}\frac{d\theta}{dX} - N^2(\theta - 1) = 0, \text{where } N^2 = \frac{2h_aL^2}{kw}$$

At X=0 i.e. at the tip of the fin $\frac{d\theta}{dX} = 0$

(2) For the wall

$$\frac{d^2\theta}{dX^2} = 0$$

A $X = 1+\delta^*$, i.e. at the surface of the wall

$$\frac{d\theta}{dX} = B_i(\theta_f - \theta)$$

At X=1 i.e. at the base of the fin

$$\theta_1 = \theta_2 \text{ and } \frac{d\theta_1}{dX} = \frac{d\theta_2}{dX}$$

To solve the above equations let us assume that $\theta - 1 = \Psi$

Then the equations and the boundary conditions become

(1) <u>For the fin</u>

$$X^{1/2}\frac{d^2\Psi}{dX^2} + \frac{X^{-1/2}}{2}\frac{d\Psi}{dX} - N^2\Psi = 0 \qquad \ldots.\ (2.2.6)$$

At the tip of the fin i.e. at $X=0$ $\frac{d\Psi}{dX} = 0$ .... (2.2.7)

(2) <u>For the wall</u>

$$\frac{d^2\Psi}{dX^2} = 0 \qquad \ldots.\ (2.2.8)$$

At the surface of the wall i.e. at $X= 1+\delta^*$

$$\frac{d\Psi}{dX} = B_i(\theta_f-1-\Psi) \qquad \ldots.\ (2.2.9)$$

At the base of the fin i.e. at $X=1$

$$\Psi_1=\Psi_2 \qquad \ldots.\ (2.2.10)$$

$$\frac{d\Psi_1}{dX} = \frac{d\Psi_2}{dX} \qquad \ldots.\ (2.2.11)$$

The boundary conditions (2.2.10) and (2.2.11) are common to both the Eqs. (2.2.6) and (2.2.8).

The solution of Eq. (2.2.6) is,

$$\Psi = X^{1/4}\left[C_1 I_{1/3}(4/3N\,X^{3/4}) + C_2 I_{-1/3}(4/3N\,X^{3/4})\right]$$

.... (2.2.12)

The use of boundary condition (2.2.7) in equation (2.2.12) requires multiplication of each of the terms of the infinite series expansion of $I_{1/3}(4/3N\ X^{3/4})$ and $I_{-1/3}(4/3N\ X^{3/4})$ by $X^{1/4}$, followed by a term-by-term differentiation. When this is performed the term involving $(\frac{d}{dX})\ (X^{1/4}I_{1/3}(4/3N\ X^{3/4})\ )$ becomes unbounded at X = 0 which necessitates that $C_1=0$.

Therefore $\Psi= X^{1/4}C_2I_{-1/3}(4/3N\ X^{3/4})$ .... (2.2.13)

The solution of equation (4.2.8) is

$$\Psi= AX +B \qquad .... (2.2.14)$$

Using the boundary condition (2.2.9) in (2.2.14), we obtain

$$A=B_i(\theta_f-1-A(1+\delta^*)-B)$$

or $A(1/B_i+1+\delta^*) = (\theta_f-1)-B$

$\therefore$ $\Psi=AX+(\theta_f-1)-A(1/B_i+1+\delta^*)$ .... (2.2.15)

Using the boundary condition (2.2.10) in Eqs. (2.2.13) and (2.2.15), we have

$$C_2I_{-1/3}(4/3N) = (\theta_f-1)-A(1/B_i+\delta^*) \qquad .... (2.2.16)$$

Similarly substituting the Boundary Condition (2.2.11) in Eqs. (2.2.13) and (2.2.15) yields

$$A= C_2\ NI_{2/3}(4/3N) \qquad .... (2.2.17)$$

Also by substituting (2.2.17) in (2.2.16), we get

$$C_2 I_{-1/3}(4/3\ N) = (\theta_f - 1) - C_2 N I_{2/3}(4/3\ N)\ (1/B_i + \delta^*)$$

or $$C_2(I_{-1/3}(4/3\ N) + N\ I_{2/3}(4/3\ N)\ (1/B_i + \delta^*)) = \theta_f - 1$$

$\therefore$ $$C_2 = \frac{\theta_f - 1}{I_{-1/3}(4/3\ N) + N I_{2/3}(4/3\ N)\ (1/B_i + \delta^*)}$$

Substituting the value of C_2 in Eq. (2.2.13), gives us

$$\Psi = \frac{X^{1/4}(\theta_f - 1) I_{-1/3}(4/3\ N\ X^{3/4})}{I_{-1/3}(4/3\ N) + N I_{2/3}(4/3\ N)\ (1/B_i + \delta^*)} \quad \text{.... (2.2.18)}$$

$\therefore$ $$\frac{d\Psi}{dX} = \frac{(\theta_f - 1)\ (N I_{2/3}(4/3\ N\ X^{3/4})\ )}{I_{-1/3}(4/3\ N) + N I_{2/3}(4/3\ N)\ (1/B_i + \delta^*)} \quad \text{.... (2.2.19)}$$

The heat transferred through the base of the fin is

$$q = KA\ \left(\frac{dT}{dX}\right)_{X=L} = KA\ \frac{T_a}{L}\ \left(\frac{d\Psi}{dX}\right)_{X=1}$$

or $$q = \frac{kwT_a}{L} \cdot \frac{(\theta_f - 1)(N I_{2/3}(4/3\ N))}{I_{-1/3}(4/3\ N) + N I_{2/3}(4/3\ N)(1/B_i + \delta^*)}$$

.... (2.2.20)

To optimize the heat transfer let us assume that the volume of the fin $\frac{2}{3}$ wL=V (a constant).

The optimization procedure is to express, q, as a function of, w, alone and then applying the condition, $\frac{dq}{dw} = 0$. The application of this condition yields the following relationship between B_i, δ^* and N.

$$\frac{1}{B_i} + \delta^* = \frac{2N^2(I_{-1/3}(4/3\ N))^2 - (I_{2/3}(4/3\ N))^2 - NI_{2/3}(4/3\ N)I_{-1/3}(4/3\ N)}{N^2(I_{2/3}(4/3\ N))^2}$$

(For derivation please refer to Appendix B)

2.3 CONCAVE PARABOLIC SPINE :

Derivation of the governing differential equation:
The generalized differential equation for any spine is given by

$$(f_2(x))^2 \frac{d^2T}{dx^2} + \frac{d}{dx}(f_2(x))^2 . \frac{dT}{dx} - \frac{2h_a}{k} f_2(x)(T-T_a)=0 \qquad \text{.... (2.3.1)}$$

The profile function $f_2(x)$ in case of the concave parabolic spine, shown in Fig. 3, is

$$f_2(x) = \left(\frac{x}{L}\right)^2 .R \qquad \text{.... (2.3.2)}$$

$$\frac{d}{dx} f_2(x) = \frac{2Rx}{L^2} \qquad \text{.... (2.3.3)}$$

Substituting Eqs. (2.3.2) and (2.3.3) in Eq. (2.3.1), we get

$$x^4 \frac{d^2T}{dx^2} + 4x^3 \frac{dT}{dx} - \frac{2h_a L^2}{kR} x^2(T-T_a) = 0 \qquad \text{.... (2.3.4)}$$

Eq. (2.3.4) is the governing differential equation for a concave parabolic spine.

The differential equation governing the flow of heat in the fin base wall is

$$\frac{d^2T}{dx^2} = 0 \qquad \text{.... (2.3.5)}$$

<u>BOUNDARY CONDITIONS</u> :

1. At the tip of the fin where $x = 0$, $\frac{dT}{dx} = 0$
2. At the wall inner surface, where $x = L+\delta$,

   $$k\frac{dT}{dx} = h_f(T_f - T)$$

3. At the base of the fin i.e. at $x=L$,

   $$T_1 = T_2 \text{ and } \frac{dT_1}{dx} = \frac{dT_2}{dx}$$

Nondimensionalizing the governing equations and boundary conditions :

Let $\theta = T/T_a$, $X = \frac{x}{L}$, $\theta_f = \frac{T_f}{T_a}$, $N^2 = \frac{2h_aL^2}{kR}$

Then

(1) For the fin, we get

$$X^4 \frac{d^2\theta}{dX^2} + 4X^3 \frac{d\theta}{dX} - N^2X^2(\theta - 1) = 0$$

At $X=0$ i.e. at the tip of the fin $\frac{d\theta}{dX} = 0$

(2) For the fin base wall

$$\frac{d^2\theta}{dX^2} = 0$$

At $X= 1+\delta^{*}$, i.e. at the surface of the wall

$$\frac{d\theta}{dX} = B_i(\theta_f - \theta)$$

At X=1, i.e. at the base of the fin

$$\theta_1 = \theta_2 \text{ and } \frac{d\theta_1}{dX} = \frac{d\theta_2}{dX}$$

To solve the above equations let us assume

$$\Psi = \theta - 1$$

Then the equations and boundary conditions become

(1) For the fin

$$X^4 \frac{d^2\Psi}{dX^2} + 4X^3 \frac{d\Psi}{dX} - N^2X^2\Psi = 0 \quad \text{.... (2.3.6)}$$

At the tip of the fin where X=0

$$\frac{d\Psi}{dX} = 0 \quad \text{.... (2.3.7)}$$

(2) For the fin base wall

$$\frac{d^2\Psi}{dX^2} = 0 \quad \text{.... (2.3.8)}$$

At $X= 1+\delta^{*}$, i.e. at the inner surface of the base wall

$$\frac{d\Psi}{dX} = B_i(\theta_f - 1 - \Psi) \quad \text{.... (2.3.9)}$$

(3) At the base of the fin, where X=1

$$\Psi_1 = \Psi_2 \quad \text{.... (2.3.10)}$$

$$\frac{d\Psi_1}{dX} = \frac{d\Psi_2}{dX} \quad \text{.... (2.3.11)}$$

The solution of Eq. (2.3.6) is

$$\Psi=C_1\, X^{P_1}+C_2\, X^{P_2} \qquad \text{.... (2.3.12)}$$

where $P_1 =(-3/2+1/2\ (9+4N^2)^{1/2})$

$P_2 =(-3/2-1/2\ (9+4N^2)^{1/2})$

The value of, C_2, in Eq. (2.3.12) must equal zero for obtaining a finite temperature excess at the tip of the fin (where X=0)

$$\therefore\quad \Psi = C_1 X^{P_1} \qquad \text{.... (2.3.13)}$$

The solution of Eq.(2.3.8) is

$$\Psi =AX+B \qquad \text{.... (2.3.14)}$$

Substituting the boundary condition (2.3.9) in Eq. (2.3.14), yields

$$B= (\theta_f-1)-A(1/B_i+\delta^*+1)$$

$$\therefore\quad \Psi=AX+(\theta_f-1)-A(1/B_i+\delta^*+1) \qquad \text{.... (2.3.15)}$$

Applying the boundary condition (2.3.10) to Eqs. (2.3.13) and (2.3.15) , we get

$$C_1= (\theta_f-1)-A(1/B_i+\delta^*) \qquad \text{.... (2.3.16)}$$

Applying the boundary condition (2.3.11) to Eqs. (2.3.13) and (2.3.15), we obtain

$$A= P_1\ C_1 \qquad \text{.... (2.3.17)}$$

Substituting the value of Eq. (2.3.17) in the Eq. (2.3.16), gives us

$$C_1 = \frac{(\theta_f - 1)}{(1 + P_1(1/B_i + \delta^*)\ )}$$

Substituting the value of, C_1, in equation (2.3.13), yields

$$\Psi = \frac{(\theta_f - 1)X^{P_1}}{(1 + P_1(1/B_i + \delta^*)\ )} \qquad \text{.... (2.3.18)}$$

$$\frac{d\Psi}{dX} = \frac{P_1 X^{P_1 - 1}(\theta_f - 1)}{(1 + P_1(1/B_i + \delta^*)\ )} \qquad \text{.... (2.3.19)}$$

Theheat transferred through the base of the fin is,

$$q = KA\left(\frac{dT}{dX}\right)_{X=L} = KA\,\frac{T_a}{L}\left(\frac{d\Psi}{dX}\right)_{X=1}$$

$$= \frac{K\pi R^2\ T_a}{L} \cdot \frac{P_1(\theta_f - 1)}{(1 + P_1(1/B_i + \delta^*))}$$

.... (2.3.20)

The procedure for optimization involves expressing q as a function of R alone and applying the condition $\frac{dq}{dR} = 0$. The optimization analysis yields the following relationship between, B_i, δ^* and N.

$$\frac{1}{B_i} = \frac{5N^2+2(9+4N^2)^{1/2}(3(9+4N^2)^{1/2}-1)}{2(9+4N^2)^{1/2}(\frac{1}{2}(9+4N^2)^{1/2}-3/2)^2}$$

(For derivation please refer to Appendix-C)

2.4 CONVEX PARABOLIC SPINE :

Derivation of the governing differential equation:

The profile function, of a convex parabolic spine (shown in Fig.4) is given by

$$f_2(x)=(\frac{x}{L})^{1/2} R \qquad \text{.... (2.4.1)}$$

$$\therefore \quad \frac{df_2(x)}{dx} = \frac{R}{2}\left(\frac{1}{Lx}\right)^{1/2} \qquad \text{.... (2.4.2)}$$

We have the general differential equation for a spine given by,

$$(f_2(x))^2 \frac{d^2T}{dx^2} + \frac{d}{dx}(f_2(x))^2 \frac{dT}{dx} - \frac{2h_a}{k} f_2(x)\ T=0 \qquad \text{.... (2.4.3)}$$

Substituting Eqs. (2.4.1) and (2.4.2) in Eq. (2.4.3), we get

$$x\frac{d^2T}{dx^2} + \frac{dT}{dx} - \frac{2h_a(Lx)^{1/2}}{kR}(T-T_a)= 0 \qquad \text{.... (2.4.4)}$$

Equation (2.4.4) is the governing differential equation for the flow of heat in a convex parabolic spine.

The equation governing the flow of heat in the spine base wall is,

$$\frac{d^2T}{dx^2} = 0 \qquad \text{.... (2.4.5)}$$

BOUNDARY CONDITIONS :

1. At the tip of the spine ($x=0$), $\frac{dT}{dx} = 0$

2. At the wall inner surface ($x=L+\delta$),

$$k\, dT/dx = h_f(T_f - T)$$

3. At the base of the spine ($x=L$),

$$T_1 = T_2 \text{ and } \frac{dT_1}{dx} = \frac{dT_2}{dx}$$

Nondimensionalization of governing equations and boundary conditions :

$$\text{Let } \theta = T/T_a,\ X = \frac{x}{L},\ \theta_f = \frac{T_f}{T_a},\ N^2 = \frac{2h_a L^2}{kR}$$

Substituting the above values in the differential equations and boundary conditions, we get

1) For the spine :

$$X\frac{d^2\theta}{dX^2} + \frac{d\theta}{dX} - N^2(X)^{1/2}(\theta - 1) = 0$$

At $X=0$, $\frac{d\theta}{dX} = 0$

2) For the spine base wall :

$$\frac{d^2\theta}{dX^2} = 0$$

At $X=1+\delta^*$, $\frac{d\theta}{dX} = B_i(\theta_f - \theta)$

3) At X=1, i.e. at the spine base

$$\theta_1=\theta_2 \text{ and } \frac{d\theta_1}{dX} = \frac{d\theta_2}{dX}$$

Let $\Psi = \theta - 1$

Then, the above equations and boundary conditions become

(1) For the spine

$$X\frac{d^2\Psi}{dX^2} + \frac{d\Psi}{dX} - N^2\Psi = 0 \qquad \text{.... (2.4.6)}$$

At X=0 i.e. at the spine tip

$$\frac{d\Psi}{dX} = 0 \qquad \text{.... (2.4.7)}$$

(2) For the spine base wall

$$\frac{d^2\Psi}{dX^2} = 0 \qquad \text{.... (2.4.8)}$$

At $X = 1+\delta^*$, $\frac{d\Psi}{dX} = B_i(\theta_f - 1 - \Psi)$ .... (2.4.9)

(3) At X=1

$$\Psi_1 = \Psi_2 \qquad \text{.... (2.4.10)}$$

and $$\frac{d\Psi_1}{dX} = \frac{d\Psi_2}{dX} \qquad \text{.... (2.4.11)}$$

The boundary conditions (2.4.10) and (2.4.11) are common to both the equations (2.4.6) and (2.4.8). The solution of equation (2.4.6) is,

$$\Psi = C_1(4/3\ N\ X^{3/4}) + C_2 k_o(4/3\ N\ X^{3/4}) \qquad \text{.... (2.4.12)}$$

For satisfying the boundary condition (2.4.7) i.e. to maintain a finite temperature excess at $X=0, C_2$ must be zero since k_o is unbounded.

$$\therefore \quad \Psi = C_1 I_o(4/3\ N\ X^{3/4}) \qquad .... (2.4.13)$$

The solution of Eq. (2.4.8) is

$$\Psi = AX+B \qquad .... (2.4.14)$$

Using the boundary condition (2.4.9), Eq. (2.4.14) can be written as

$$B = (\theta_f - 1) - A\ (1/B_i + 1 + \delta^*)$$

Substituting the value of B in Eq. (2.4.14), we get

$$\Psi = AX + (\theta_f - 1) - A(1/B_i + 1 + \delta^*) \qquad .... (2.4.15)$$

Applying the boundary condition (4.2.10) to both the Eqs. (2.4.13) and (2.4.15), we obtain

$$C_1\ I_o(4/3\ N) = (\theta_f - 1) - A(1/B_i + \delta^*) \qquad .... (2.4.16)$$

Using the boundary condition (2.4.11) in the Eqs. (2.4.13) and (2.4.15) yields,

$$A = C_1 N . I_1(4/3\ N) \qquad .... (2.4.17)$$

Substituting Eq. (2.4.17) in Eq. (2.4.16), we can find out the value of C_1

$$C_1 . I_o(4/3\ N) = (\theta_f - 1) - C_1 N.\ I_1(4/3\ N).\ k$$

$$\therefore\quad C_1 = \frac{(\theta_f - 1)}{I_o(4/3\ N) + NkI_1(4/3\ N)}$$

where $\quad k = \frac{1}{B_i} + \delta^*$

Substituting the value of C_1 in Eq. (2.4.13), gives us

$$\Psi = \frac{(\theta_f - 1) I_o(4/3\ N\ X^{3/4})}{I_o(4/3\ N) + NkI_1(4/3\ N)} \qquad \text{.... (2.4.18)}$$

$$\frac{d\Psi}{dX} = \frac{X^{-1/4}\ N\ I_1(4/3\ N\ X^{3/4})(\theta_f - 1)}{I_o(4/3\ N) + N.k.I_1(4/3\ N)} \qquad \text{.... (2.4.19)}$$

The heat transferred through the spine base is

$$q = KA\left(\frac{dT}{dX}\right)_{X=L}$$

$$= \frac{KAT_a}{L}\left(\frac{d\Psi}{dX}\right)_{X=1}$$

$$\therefore\quad q = \frac{k\pi R^2 T_a(\theta_f - 1) N.I_1(4/3\ N)}{L(I_o(4/3\ N) + I_1(4/3\ N).k.N.)} \qquad \text{.... (2.4.20)}$$

Applying the optimization criterion, $\frac{dq}{dR} = 0$ leads to the following relationship between B_i, δ^* and N

$$1/B_i = \frac{5}{3} . \frac{(I_o(4/3\ N))^2}{(I_1(4/3\ N))^2} - \frac{5}{3} - \frac{2}{N}\frac{I_o(4/3\ N)}{I_1(4/3\ N)} - \delta^*$$

(For derivation please refer to Appendix-D)

CHAPTER-III

RESULTS AND DISCUSSIONS

3.1 RESULTS AND DISCUSSIONS :

3.11 .LONGITUDINAL FINS OF CONCAVE PARABOLIC PROFILE :

Figure (5) depicts the optimum dimensions for concave parabolic fins. It represents the relationship, between N and Bi that must be satisfied by a fin to transfer the maximum amount of heat. It is seen from the figure that as N increases the value of 1/Bi decreases, for a particular δ^* value. For a higher value of N, a higher Bi is necessary to transf the same amount of heat. Therefore 1/Bi decreases as N increases. As δ^* increases the curves become more steeper and for a constant value of N, 1/Bi decreases. When the thicknes of the fin base wall is more, the resistance offered to the flow of heat by the wall is greater and so heat transferred is less. To compensate this reduction in heat transmission a higher Bi value is required.

Figure (6) shows the variation of the fin efficiency(η) with the parameter N for constant values of $(1/Bi+\delta^*)$. As the value of N increases the value of the efficiency decreases. It is observed from the figure that as $(1/Bi+\delta^*)$ increases, the curves become steeper as they shift diagonally to the left and for a constant value of N, the efficiency decreases.

The top most curve, in which the value of $(1/B_i+\delta^*)$ is equal to zero, represents the simplest case when the base temperature of the fin and the fluid temperature are equal. This corresponds to the situation where the heat transfer coefficient, h_f, between the fluid and the fin base wall is infinite, and the fin base wall resistance is zero. For higher values of $(1/B_i+\delta^*)$, h_f becomes finite and therefore introduces a resistance to the flow of heat from the fluid to the fin base wall, in addition to the resistance of fin base wall itself, which results in a decreased efficiency.

3.1.2 LONGITUDINAL FIN OF CONVEX PARABOLIC PROFILE.

Figure (7) is the plot of the optimum dimensions for the convex parabolic fins. It is a graph of $1/B_i$ vs N for various values of δ^*. As N increases $1/B_i$ decreases. For higher values of δ^*, the decrease in $1/B_i$ becomes steeper and shifts towards lower N.

Figure (8) gives the variation of the efficiency of convex parabolic fins with N. It is seen that as N increases the efficiency decreases for various δ^* values. As δ^* increases the curves become steeper and shift diagonally towards lower values of N and for a constant N the efficiency suffers a reduction in magnitude. This is, as before, due to the reduction in heat transfer caused by the increased resistance of the fin base wall.

3.1.3 CONCAVE PARABOLIC SPINES :

Figure (9) shows the optimum dimensions for concave parabolic spines, for various δ^* values. This 1/Bi vs N plot represents the relationship between 1/Bi and N that has to be satisfied to transfer the maximum amount of heat through the spine. As N increases 1/Bi decreases, for a particular δ^* value. Curves become steeper as δ^* increases giving rise to a larger drop in 1/Bi with N.

Figure (10) presents the change in the efficiency of concave parabolic spines with N, for different values of $(1/Bi + \delta^*)$. Efficiency is seen as usual to decrease with increasing N, for a particular $(1/Bi+\delta^*)$ value. It can also be observed from the figure that a decrease in efficiency occurs as a result of increasing $(1/Bi+\delta^*)$ value, for a constant value of N.

3.1.4 CONVEX PARABOLIC SPINES :

Figure (11) exhibits the effect of δ^* on the optimum dimensions of a convex parabolic spine. It is a graph of 1/Bi Vs. N, for different δ^* values. For a given value of δ^*, 1/Bi is seen to decrease with increasing N. An increase in the value of δ^* marks a decrease in 1/Bi for a constant value of N.

Figure (12) is the plot of efficiency Vs. N. As is the case before, efficiency decreases with increasing N, for a constant value of $(1/B_i+\delta^*)$. As $(1/B_i+\delta^*)$ increases, the efficiency decreases for a particular value of N.

Figures 13,14,15 and 16 give the optimum dimensions of various types of fins for $\delta^*=0$, 1,2, and 3, respectively. These are plots of $1/B_i$ Vs. N. For a constant value of N less than, 1, the concave parabolic spine gives a higher value of $1/B_i$, compared to all other analysed fin types. This means that for a given N value below 1, the optimal length required for the concave parabolic spine is going to be the least compared to the other analysed fin configurations which yield lower $1/B_i$ values.

Figures 17,18,19 and 20 depict the variation of the efficiency of various fins with the fin parameter N, for $(1/B_i+\delta^*)=$ 0,1,2, and 3, respectively. It is seen that the concave parabolic spine has the highest efficiency and concave parabolic fin the lowest. The efficiency of other fins lies in between these two. The efficiency of a particular fin geometry is seen to increase with decreasing N values.

Figure 21 gives the optimum dimensions of various fins analysed in the present work and in a previous one (Ref.10). It is a plot of $1/B_i$ Vs N for $\delta^*=0$.

Figure 22 is a plot of efficiency Vs. N for $1/B_i+\delta^*=0$ for various fin profiles, present and previous (Ref.10). As can be seen from the graph concave parabolic spine is the most efficient of all the fin types considered.

CONCLUSIONS

3.2. THE MAIN CONCLUSIONS DRAWN FROM THIS STUDY ARE AS FOLLOWS :

1. The efficiency of all the fins investigated herein decreases as the fin parameter N increases.

2. For a constant value of fin parameter N, the efficiency decreases as the value of $(1/Bi+\delta^*)$ increases.

3. The value of 1/Bi decreases as the fin parameter, N, increases.

4. For a constant value of N, 1/Bi decreases as δ^* increases.

5. Out of the four types of fins analysed the concave parabolic spine is the most efficient and the concave parabolic fin the least efficient.

6. Under the optimum condition , the concave parabolic spine yields a maximum value of 1/Bi, for a constant value of N, while the convex parabolic spine gives a minimum value.

7. The parameters Bi and δ^* have a considerable effect on the optimum dimensions of the fins.

3.3 SUGGESTIONS FOR FUTURE WORK.

A similar analysis can be carried out for other types of fins like, longitudinal fins of trapezoidal profile, truncated conical spine, radial fins of concave parabolic, convex parabolic, hyperbolic, triangular profiles etc.

In the present study constant fluid and fin material properties have been assumed. Thermal dependence of these fluid and fin material properties could also be considered in a future optimization analysis.

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APPENDIX-A

From Eq. (2.1.20) on page (12)

We see that

$$q \;\alpha\; \frac{P_1 w^2}{1+P_1(1/B_i+\delta^*)}$$

The optimization criterion is $\frac{dq}{dw} = 0$.

$$\therefore\; (1+P_1(1/B_i+\delta^*)).(P_1.2w+w^2((4N^2+1)^{-1/2}\,2N(\tfrac{-3N}{2w})\;))$$

$$= P_1w^2(P_1(\frac{1/B_i+\delta^*}{w})+(1/B_i+\delta^*)(4N^2+1)^{-1/2}\,2N(\tfrac{-3N}{2w})\;)$$

.... (2.1.21)

$\because$ (i) $N^2 = \frac{2h_aL^2}{kw} = \frac{18h_aV^2}{kw^3}$

where $V = \frac{1}{3}Lw$

$\therefore\; \frac{dN}{dw} = \frac{-3N}{2w}$

(ii) $\frac{d}{dw}(1/B_i+\delta^*) = \frac{d}{dw}(\frac{k}{h_fL} + \frac{\delta}{L})$

$$= \frac{d}{dw}(\frac{kw}{3h_fV} + \frac{\delta w}{3V}$$

$$= \frac{k}{(3h_fV} + \frac{\delta}{3V}) = \frac{1}{B_iw} + \frac{\delta^*}{w}$$

$$=(\frac{1/B_i+\delta^*}{w})$$

LHS. of Eq. (2.1.21) =

$$(1+P_1(1/B_i+\delta^*))(P_1 2w+w^2(1+4N^2)^{-1/2}(\frac{-3N^2}{w}))$$

$$= (1+P_1(1/B_i+\delta^*))(P_1 2w+\frac{w^2}{w}(-3N^2(1+4N^2)^{-1/2}))$$

$$=(1+P_1(1/B_i+\delta^*))(w(2P_1-3N^2(1+4N^2)^{-1/2}))$$

RHS of Eq. (2.1.21)

$$P_1\frac{w^2}{1w}\ (P_1(1/B_i+\delta^*)+(1/B_i+\delta^*).(1+4N^2)^{-1/2}(-3N^2))$$

$$= P_1w(1/B_i+\delta^*)(P_1-3N^2(1+4N^2)^{-1/2})$$

Eq. (2.1.21) can now be written as,

$$(1+P_1(1/B_i+\delta^*))w\ (2P_1-3N^2(1+4N^2)^{-1/2})$$

$$= P_1\ w(1/B_i+\delta^*)(P_1-3N^2(1+4N^2)^{-1/2})$$

$$\therefore\quad 1+P_1(1/B_i+\delta^*)(2P_1-3N^2(1+4N^2)^{-1/2}=P_1(1/B_i+\delta^*)$$

$$x(P_1-3N^2(1+4N^2)^{-1/2})$$

$$2P_1+2P_1^2(1/B_i+\delta^*)-3N^2(1+4N^2)^{-1/2}-P_1(1/B_i+\delta^*)\ 3N^2(1+4N^2)^{-1/2}$$

$$= P_1^2(1/B_i+\delta^*)-P_1(1/B_i+\delta^*)3N^2(1+4N^2)^{-1/2}$$

$$\therefore\quad 2P_1+P_1^2(1/B_i+\delta^*)-3N^2(1+4N^2)^{-1/2}=0$$

$$\therefore\quad P_1^2(1/B_i+\delta^*)=3N^2(1+4N^2)^{-1/2}-2P_1$$

$$\therefore\quad (1/B_i+\delta^*)=\frac{3N^2(1+4N^2)^{-1/2}-2P_1}{P_1}$$

Replacing P_1 by the expression, $P_1= -1/2+1/2\ (1+4N^2)^{1/2}$ in the above equation yields

$$(1/Bi+\delta^{*}) = \frac{3N^{2}(1+4N^{2})^{-1/2} + 1 - (1+4N^{2})^{1/2}}{(-1/2+1/2\ (1+4N^{2})^{1/2})^{2}}$$

$$= \frac{3N^{2} + (1+4N^{2})^{1/2} - (1+4N^{2})}{(1+4N^{2})^{1/2}(1/4+1/4\ (4N^{2}+)^{1/2} - 1/2(1+4N^{2})^{1/2})}$$

$$= \frac{4\ (3N^{2}-4N^{2}+1+(1+4N^{2})^{1/2})}{(1+4N^{2})^{1/2}((1+4N^{2})^{1/2}-1)^{2}}$$

$$\therefore \quad \frac{1}{Bi} + \delta^{*} = \frac{4(\ (1+4N^{2})^{1/2} - N^{2} - 1\ )}{(1+4N^{2})^{1/2}((1+4N^{2})^{1/2}-1)^{2}}$$

$$\therefore \quad \frac{1}{Bi} = \frac{4(\ (1+4N^{2})^{1/2} - N^{2} - 1)}{(1+4N^{2})^{1/2}(\ (1+4N^{2})^{1/2}-1)^{2}} - \delta^{*}$$

APPENDIX-B

From Eq. (2.2.20) on page number (17) we see that

$$q \ \alpha \ \frac{w^2(NI_{2/3}(4/3\ N)}{I_{-1/3}(4/3\ N)+NI_{2/3}(4/3\ N)(1/B_i+\delta^*)}$$

Application of the optimization condition $\frac{dq}{dw}=0$ yields,

$$(I_{-1/2}(4/3\ N)+NI_{2/3}(4/3\ N)(1/B_i+\delta^*))\frac{d}{dw}(w^2NI_{2/3}(4/3\ N))$$

$$= (w^2NI_{2/3}(4/3\ N))\frac{d}{dw}(I_{-1/3}(4/3\ N)+NI_{2/3}(4/3\ N)(1/B_i+\delta^*)) \quad \text{.... (2.2.21)}$$

where

$$\text{(i)} \quad \frac{d}{dw}\ I_{2/3}(4/3\ N) = I_{-1/3}(4/3\ N)-\frac{2}{3\frac{4}{3}N}\ I_{2/3}(4/3\ N)\frac{4}{3}(\frac{-3N}{2w})$$

$$\left(\text{since } \frac{dN}{dw}=\frac{-3N}{2w}\right)$$

$$= -I_{-1/3}(4/3\ N)-\frac{1}{2N}I_{2/3}(4/3\ N)\ \frac{2N}{w}$$

$$= \frac{2N}{w}\ 1/2NI_{2/3}(4/3\ N)-I_{-1/3}(4/3\ N)$$

$$\therefore \quad N\frac{d}{dw}\ I_{2/3}(4/3\ N)= \frac{2N^2}{w}\ \frac{1}{2N}\ I_{2/3}(4/3\ N)-I_{-1/3}(4/3\ N)$$

(ii) $\frac{d}{dw} \quad I_{-1/3}(4/3\ N) \quad = I_{-4/3}(4/3\ N) + \frac{1}{3\frac{4}{3}N} \ I_{-1/3}(4/3\ N)$

$$\frac{4}{3}(-\frac{3N}{2w})$$

$$= - I_{-4/3}(4/3\ N) + \frac{1}{4N} I_{-1/3}(4/3\ N)\frac{2N}{w}$$

$$= -I_{2/3}(4/3\ N) - \frac{1}{2N} I_{-1/3}(4/3\ N) + \frac{1}{4N} I_{-1/3}$$

$$(4/3\ N)\ \frac{2N}{w}$$

$$= \frac{1}{4N}\ I_{-1/3}(4/3\ N) - I_{2/3}(4/3\ N)\frac{2N}{w}$$

$$\therefore \frac{d}{dw}\ I_{-1/3}(4/3\ N) = \frac{1}{2w}\ I_{-1/3}(4/3\ N) - \frac{2N}{w}\ I_{2/3}(4/3\ N)$$

substituting the above values into Eq. (2.2.1) we get

L.H.S. of (2.2.21)

$$(I_{-1/3}(4/3\ N) + NI_{2/3}(4/3\ N)(1/B_i + \delta^*))(NI_{2/3}(4/3\ N)\ 2w + w^2$$

$$I_{2/3}(4/3\ N)(-\frac{3N}{2w}) + N\frac{d}{dw}(I_{2/3}(4/3\ N))$$

$$= I_{-1/3}(4/3\ N) + NI_{2/3}(4/3\ N)(1/B_i + \delta^*)2w\ NI_{2/3}(4/3\ N) -$$

$$\frac{3Nw}{2}\ I_{2/3}(4/3N) + NwI_{2/3}(4/3\ N) - 2N^2w\ I_{1/3}(4/3\ N)$$

$$= I_{-1/3}(4/3\ N) + NI_{2/3}(4/3N)(1/B_i + \delta^*) - 3/2Nw\ I_{2/3}(4/3\ N) -$$

$$2N^2wI_{-1/3}(4/3\ N)$$

$$= 3/2NwI_{-1/3}(4/3\ N)I_{2/3}(4/3\ N)+ \frac{3}{2}(1/B_i+\delta^*)N^2w\ I_{2/3}(4/3\ N)^2$$
$$-2N^2w(I_{-1/3}(4/3\ N))^2-2KN^3w\ I_{2/3}(4/3\ N)I_{-1/3}(4/3\ N)$$

Also R.H.S. of Eq. (2.2.21)

$$(w^2(NI_{2/3}(4/3\ N))(\frac{d}{dw}(I_{-1/3}(4/3\ N))+(1/B_i+\delta^*)(I_{2/3}(4/3\ N)(\frac{-3N}{2w})$$
$$+\frac{2N^2}{w}(\frac{1}{2N}I_{2/3}(4/3\ N)-I_{-1/3}(4/3\ N))$$
$$+ NI_{2/3}(4/3\ N)\ (\frac{1/B_i+\delta^*}{w}))$$

$$= w^2(NI_{2/3}(4/3\ N))\frac{1}{2w}I_{-1/3}(4/3\ N)- \frac{2N}{w}I_{2/3}(4/3\ N)-(1/B_i+\delta^*)$$
$$I_{2/3}(4/3\ N)\frac{3N}{2w} + \frac{(\frac{1}{B_i}+\delta^*)}{w}I_{2/3}(4/3\ N)-2N^2\frac{(1/B_i+\delta^*)}{w}$$
$$I_{1-3}(4/3\ N)+ \frac{(1/B_i+\delta^*)}{w}NI_{2/3}(4/3\ N)$$

$$= (Nw^2\ I_{2/3}(4/3\ N))\ (1/2\ \frac{(1/B_i+\delta^*)}{w}\ NI_{2/3}(4/3\ N)- \frac{2N}{w}I_{2/3}(4/3\ N)$$
$$+ \frac{1}{2w}\ I_{-1/3}(4/3\ N)- \frac{2N^2(1/B_i+\delta^*)}{w}I_{-1/3}(4/3\ N))$$

$$= 1/2(1/B_i+\delta^*)N^2w(I_{2/3}(4/3\ N))^2-2N^2w(I_{2/3}(4/3\ N))^2$$
$$+ \frac{Nw}{2}I_{2/3}(4/3\ N)I_{-1/3}(4/3\ N)-2N^3w\ I_{2/3}(4/3\ N)\ I_{-1/3}(4/3N)(1/B_i+\delta^*)$$

Equating L.H.S. and R.H.S. of Eq. (2.2.21) yields

$$3/2Nw\, I_{-1/3}(4/3\ N)I_{2/3}(4/3\ N)+ \frac{3}{2}(1/B_i+\delta^*)N^2w(I_{2/3}(4/3\ N)\ )^2$$
$$-2N^2w(I_{-1/3}(4/3\ N))^2-2(1/B_i+\delta^*)N^3w\ I_{2/3}(4/3\ N)I_{-1/3}(4/3\ N)$$
$$= 1/2(1/B_i+\delta^*)N^2w(I_{2/3}(4/3\ N))^2-2N^2w(I_{2/3}(4/3\ N))^2$$
$$+ \frac{Nw}{2}\ I_{2/3}(4/3\ N)I_{-1/3}(4/3\ N)-(2N^3w(1/B_i+\delta^*)I_{2/3}(4/3\ N)$$
$$+I_{-1/3}(4/3\ N)\ ) \qquad \ldots. \ (2.2.22)$$

Rearranging the terms in Eq. (2.2.22), gives

$$Nw\ I_{2/3}(4/3\ N)I_{-1/3}(4/3\ N)+(1/B_i+\delta^*)N^2w(I_{2/3}(4/3\ N)\ )^2$$
$$-\ 2N^2w\ ((I_{-1/3}(4/3\ N))^2-(I_{2/3}(4/3\ N))^2\ ) = 0$$

$$\therefore\quad (1/B_i+\delta^*)N^2w(I_{2/3}(4/3\ N)\ )^2$$

$$= 2N^2w(I_{-1/3}(4/3\ N))^2-(I_{2/3}(4/3\ N)^2 -NwI_{2/3}(4/3\ N)I_{-1/3}(4/3\ N)$$

$$\therefore\quad \frac{1}{B_i}+\delta^* = \frac{2N^2(I_{-1/3}(4/3\ N))^2-(I_{2/3}(4/3\ N))^2-NI_{2/3}(4/3\ N)I_{-1/3}(4/3N)}{N^2(I_{2/3}(4/3\ N)\ )^2}$$

$$\therefore\quad \frac{1}{B_i} = \frac{2N^2((I_{-1/3}(4/3\ N))^2-(I_{2/3}(4/3\ N))^2)-NI_{2/3}(4/3\ N)I_{-1/3}(4/3N)}{N^2(I_{2/3}(4/3N)\ )^2}-\delta^*$$

APPENDIX-C

From equation (2.3.20) on page number (22)

$$q\alpha \frac{R^4P_1}{(1+P_1(1/B_i+\delta^*)}$$

Applying the condition $\frac{dq}{dR}=0$, we get

$$(1+P_1k)\frac{d}{dR}(R^4P_1)=R^4P_1(1+P_1k) \qquad \text{.... (2.3.21)}$$

where $P_1=(\frac{-3}{2}+\frac{1}{2}(9+4N^2)^{1/2})$ $k=(\frac{1}{B_i}+\delta^*)$

L.H.S. of Eq. (2.3.21)

$$= (1+P_1k)(4R^3P_1+R^4\frac{d}{dw}(-3/2+1/2(9+4N^2)^{1/2}))$$

$$= (1+P_1k)(4R^3P_1+R^4(-\frac{5N^2}{R}(9+4N^2)^{-1/2}))$$

Since $\frac{dN}{dR}=\frac{-5N}{2R}$

$$= (1+P_1k)(4R^3P_1-5N^2R^3(9+4N^2)^{-1/2})$$

$$= 4R^3P_1+4R^3P_1^2k-5N^2R^3(9+4N^2)^{-1/2}-5N^2R^3P_1k(9+4N^2)^{-1/2}$$

Also R.H.S. of Eq. (2.3.21)

$$= R^4P_1(P_1\frac{2k}{R}+k(\frac{-5N^2}{R}(9+4N^2)^{-1/2}$$

Since $\frac{dk}{dR}=\frac{2k}{R}$

$$= R^4P_1(\frac{2P_1k}{R}-\frac{5N^2k}{R}(9+4N^2)^{-1/2})$$

$$= 2P_1^2kR^3-5N^2kP_1R^3(9+4N^2)^{-1/2}$$

Eq. (2.3.21) therefore becomes.

$$4R^3P_1+4R^3P_1^2k-5N^2R^3(9+4N^2)^{-1/2}-2P_1^2\ kR^3=0$$

$$k(2P_1^2R^3)=5N^2R^3(9+4N^2)^{-1/2}-4R^3P_1$$

$$k=\frac{5N^2(9+4N^2)^{-1/2}-4P_1}{2P_1^2} \qquad \text{.... (2.3.22)}$$

Back-substitution of the values of k and P_1 into Eq. (2.3.22) gives

$$1/B_i+\delta^*=\frac{5N^2(9+4N^2)^{-1/2}-4P_1}{2P_1^2}$$

$$=\frac{5N^2(9+4N^2)^{-1/2}-4(-3/2+1/2(9+4N^2)^{1/2})}{2(-3/2+1/2(9+4N^2)^{1/2})^2}$$

$$\therefore\quad 1/B_i=\frac{5N^2+2(9+4N^2)^{1/2}(3(9+4N^2)^{1/2}-1)}{2(9+4N^2)^{1/2}(1/2(9+4N^2)^{1/2}-3/2)^2}-\delta^*$$

APPENDIX-D

From equation (2.4.20) on page number (27), we have,

$$q\alpha \frac{NR^4\ I_1(4/3N)}{I_o\ (4/3N)\ +\ KNI\ \ (4/3N)}$$

Applying the optimization criterion $\frac{dq}{dR} = 0$, we get

$$(I_o\ (4/3N) + KNI_1\ (4/3N))\ \frac{d}{dR}\ (NR^4 .\ I_1\ (4/3N))$$

$$= NR^4 I_1\ (4/3N) .\ \frac{d}{dR}\ (I_o(4/3N) + KNI_1(4/3N))$$

$$\text{.....}\ (2.4.21)$$

L.H.S. of Eq. (2.4.21)

$$= (I_o(4/3N)+KNI_1(4/3N)) \cdot (NR^4 (\ (I_o(4/3N) - \frac{3}{4N}\ I_1(4/3N)\quad .\ \frac{4}{3}\ (\frac{-5N}{2R})\ )$$

$$+\ I_1(4/3N)\quad 4NR^3 + R^4(\frac{-5N}{2R}\ ))\,)$$

$$= (I_o(4/3N)+KNI_1(4/3N)) \cdot (NR^4\ (\ \frac{5}{2R}\ I_1(4/3N) - \frac{10N}{3R}\ I_o(4/3N)\ )$$

$$+ I_1(4/3N)\ (\ 4NR^3 - 5/2NR^3\ ))$$

$$= (I_o(4/3N)\ +\ KNI_1(4/3N))\quad (4NR^3 I_1(4/3N)\ -\ \frac{10}{3}\ N^2R^3 I_o(4/3N)\ )$$

$$=\ 4NR^3 I_o(4/3N)\ I_1(4/3N)\ +\ 4kN^2R^3\ \ (I_1(4/3N)\ )^2$$

$$-\ \frac{10}{3}\ N^2R^3\ (I_o(4/3N)\ )^2 ,\ \frac{10}{3}\ kN^3R^3 I_o(4/3N) I_1(4/3N)$$

and R.H.S. of Eq.(2.4.21)

$$= NR^4(I_1(4/3\ N))\ ((-\frac{10N}{3R})I_1(4/3\ N)+I_1(4/3\ N)\frac{2NK}{R}-\frac{5NK}{2R})$$

$$+ kN((I_o(4/3\ N)-\frac{3}{4N}I_1(4/3\ N))(-\frac{10N}{3R}))$$

$$= NR^4I_1(4/3\ N)(-\frac{10N}{3R}I_1(4/3\ N)+\frac{2NK}{R}I_1(4/3\ N)-\frac{5NK}{2R}I_1(4/3\ N)$$

$$+\frac{5NK}{2R}I_1(4/3\ N)-\frac{10KN^2}{3R}I_o(4/3\ N))$$

$$= \frac{-10}{3}N^2R^3(I_1(4/3\ N))^2+2N^2R^3k(I_1(4/3\ N))^2-\frac{10}{3}kN^3R^3I_o$$

$$(4/3\ N)I_1(4/3\ N)$$

Equating L.H.S. and R.H.S., Equation (2.4.21) can now be written as,

$$4NI_o(4/3N)I_1(4/3\ N)+(4/3\ N)+4kN^2(I_1(4/3N))^2-\frac{10}{3}N^2(I_o(4/3\ N))^2$$

$$+\frac{10}{3}N^2(I_1(4/3\ N))^2-2N^2k(I_1(4/3\ N))^2=0$$

$$\therefore\ k = \frac{\frac{10}{3}N^2((I_o(4/3\ N))^2-(I_1(4/3\ N))^2)-4NI_o(4/3\ N)I_1(4/3\ N)}{2N^2(I_1(4/3\ N))^2}$$

substituting $k = 1/B_i+\delta^*$, yields

$$\frac{1}{B_i} = \frac{5}{3}\frac{(I_o(4/3\ N))^2}{(I_1(4/3\ N))^2}-\frac{5}{3}-\frac{2}{N}\frac{I_o(4/3\ N)}{I_1(4/3\ N)}-\delta^*$$

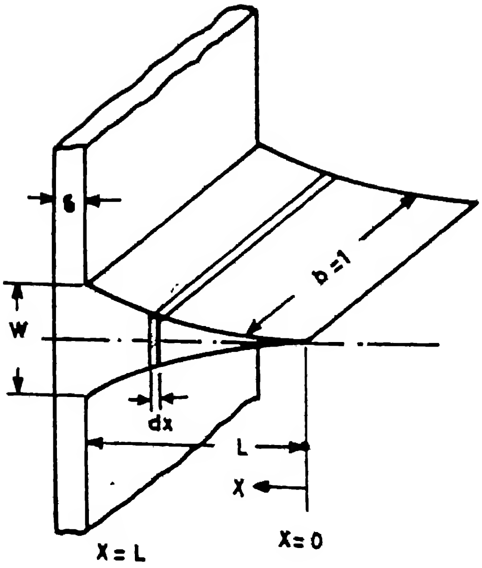

FIG.1 (a) LONGITUDINAL FIN OF CONCAVE PARABOLIC PROFILE.

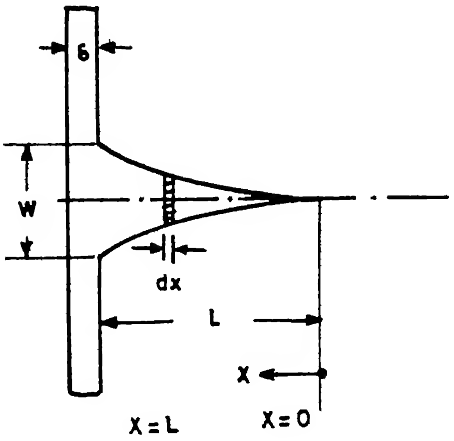

FIG.1(b) CROSS SECTIONAL VIEW .

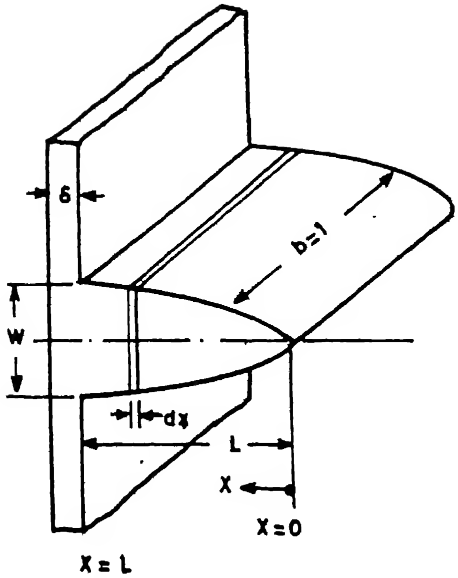

FIG. 2(a) LONGITUDINAL FIN OF CONVEX PARABOLIC PROFILE.

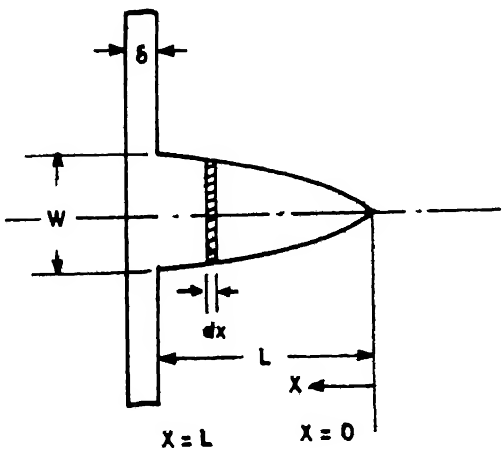

FIG. 2(b) CROSS SECTIONAL VIEW.

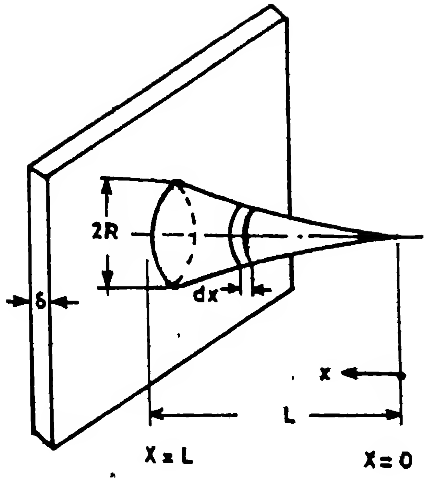

FIG. 3(a) SPINE OF CONCAVE PARABOLIC PROFILE.

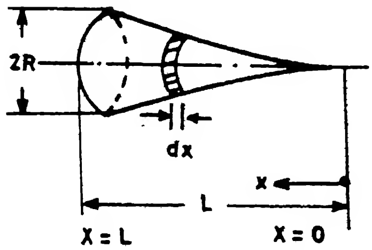

FIG. 3 (b) CROSS SECTIONAL VIEW.

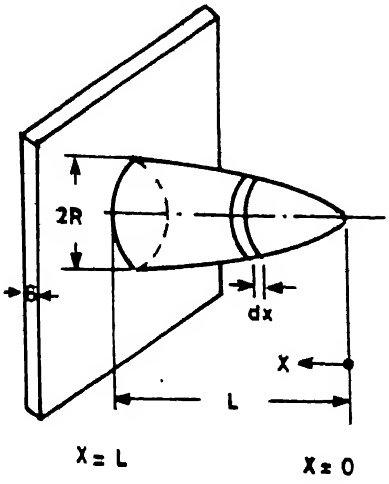

FIG.4(a) SPINE OF CONVEX PARABOLIC PROFILE.

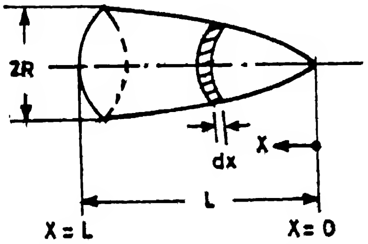

FIG. 4(b) CROSS SECTIONAL VIEW .

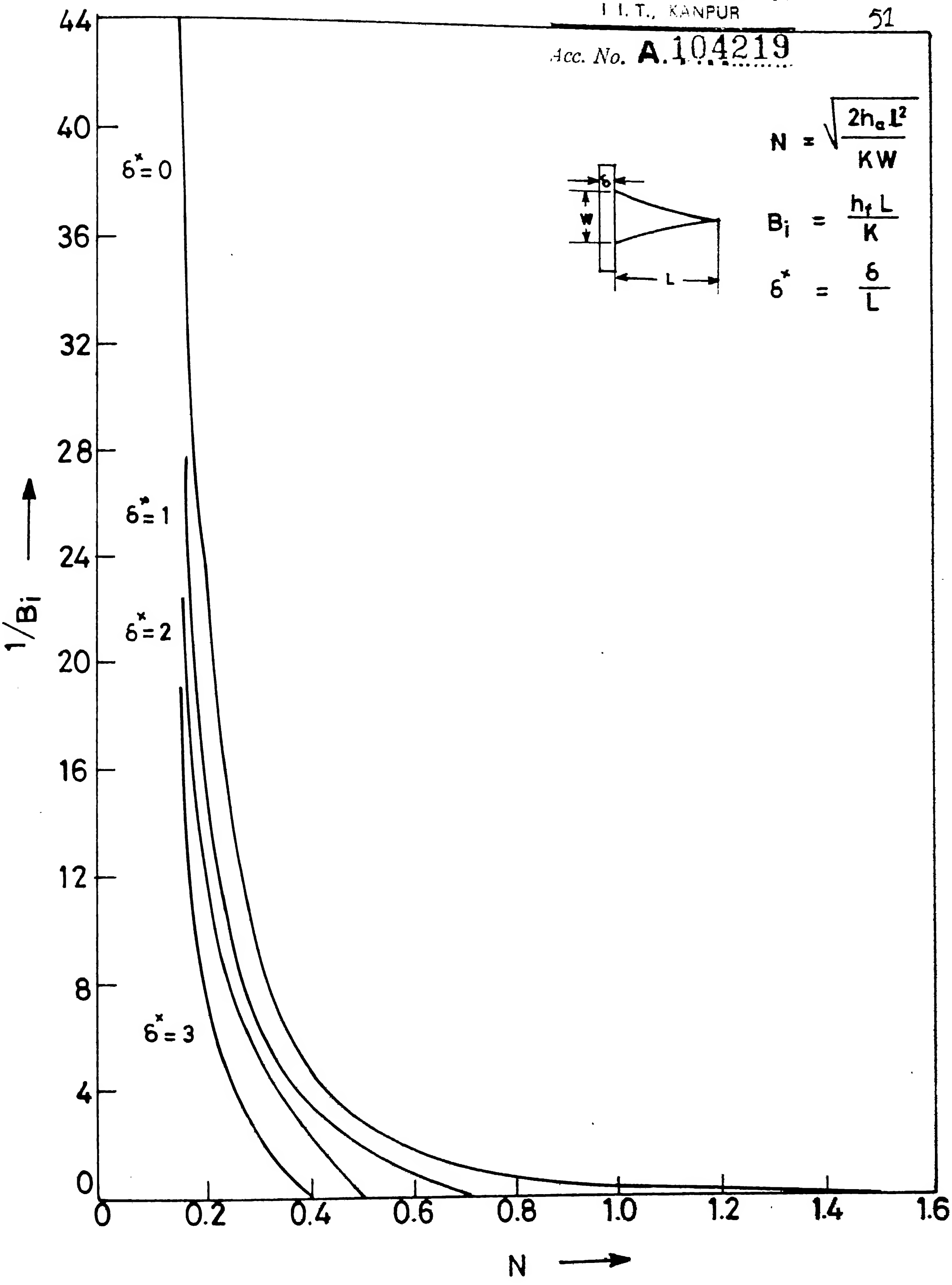

FIG. 5 OPTIMUM DIMENSIONS FOR CONCAVE PARABOLIC FIN.

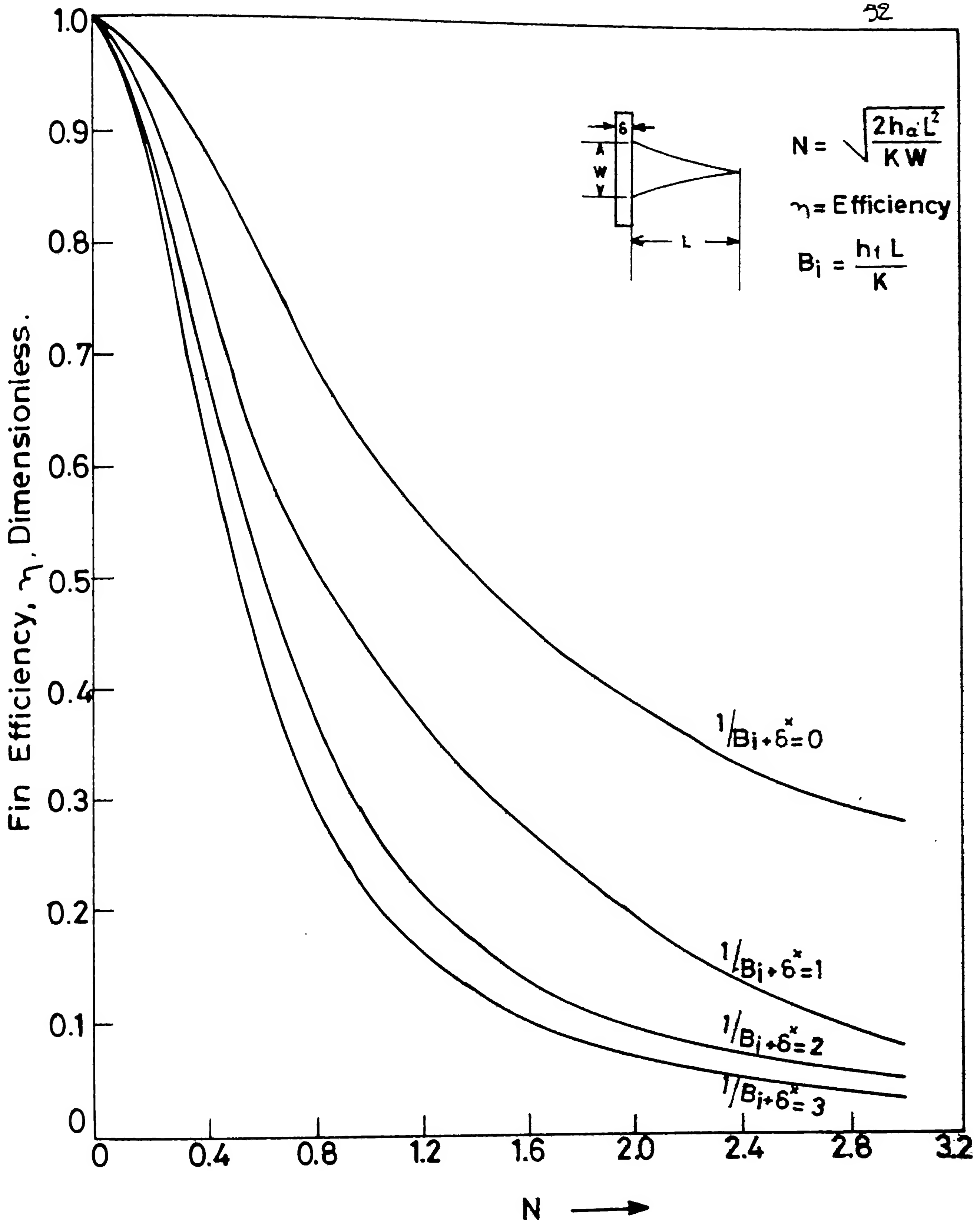

FIG. 6. FIN EFFICIENCY OF CONCAVE PARABOLIC FIN.

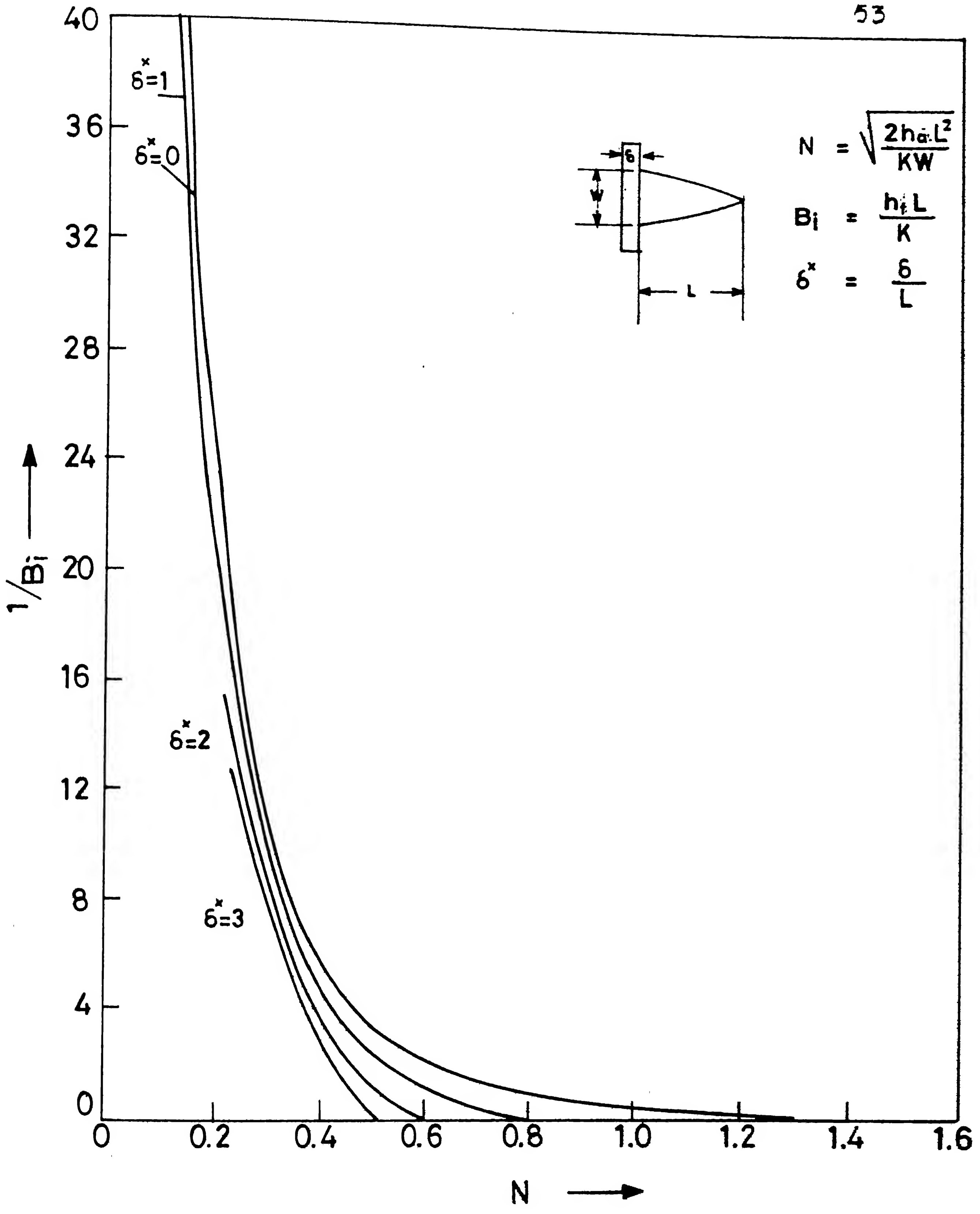

FIG. 7 OPTIMUM DIMENSIONS FOR CONVEX PARABOLIC FIN.

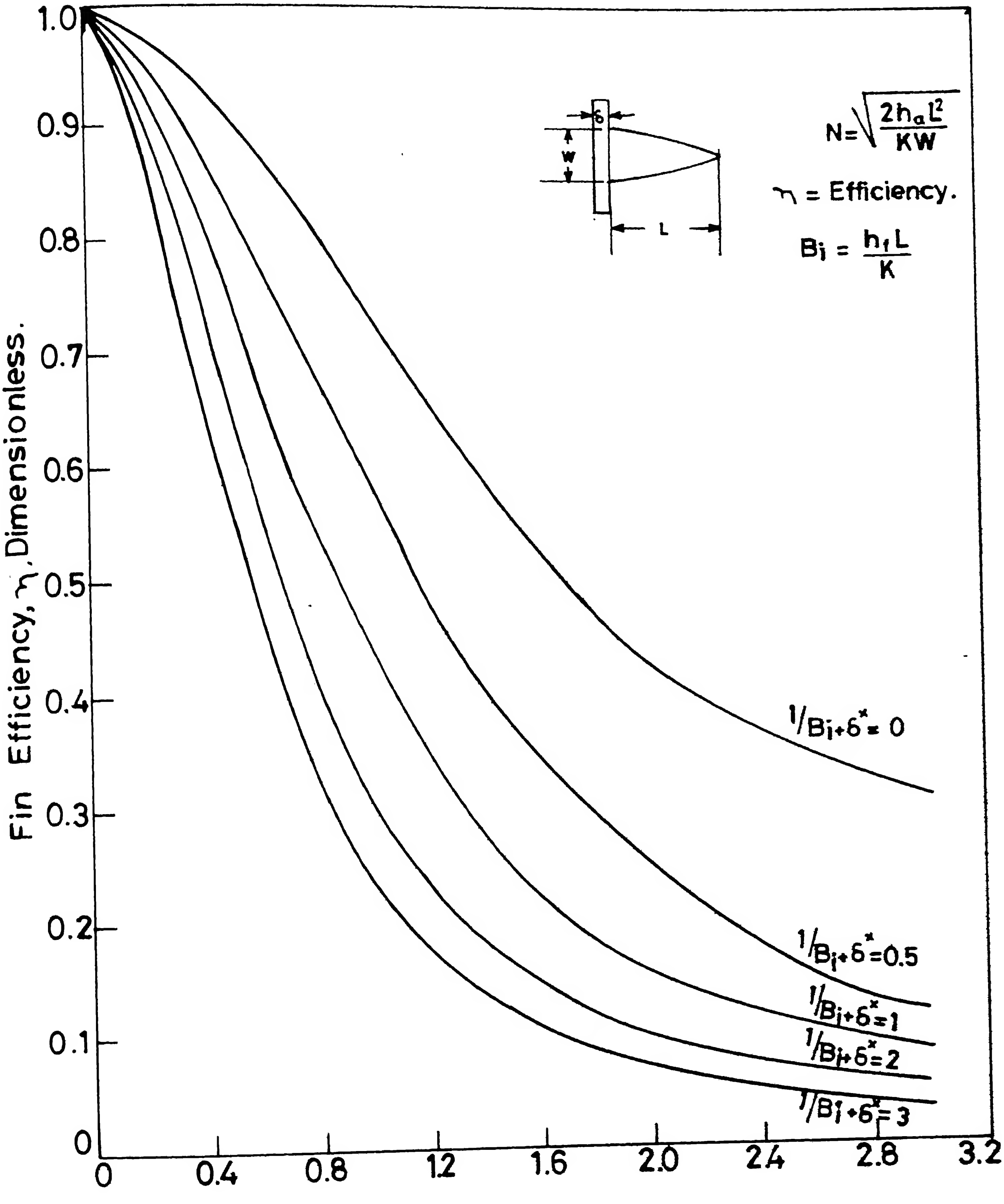

FIG. 8 FIN EFFICIENCY OF CONVEX PARABOLIC FIN.

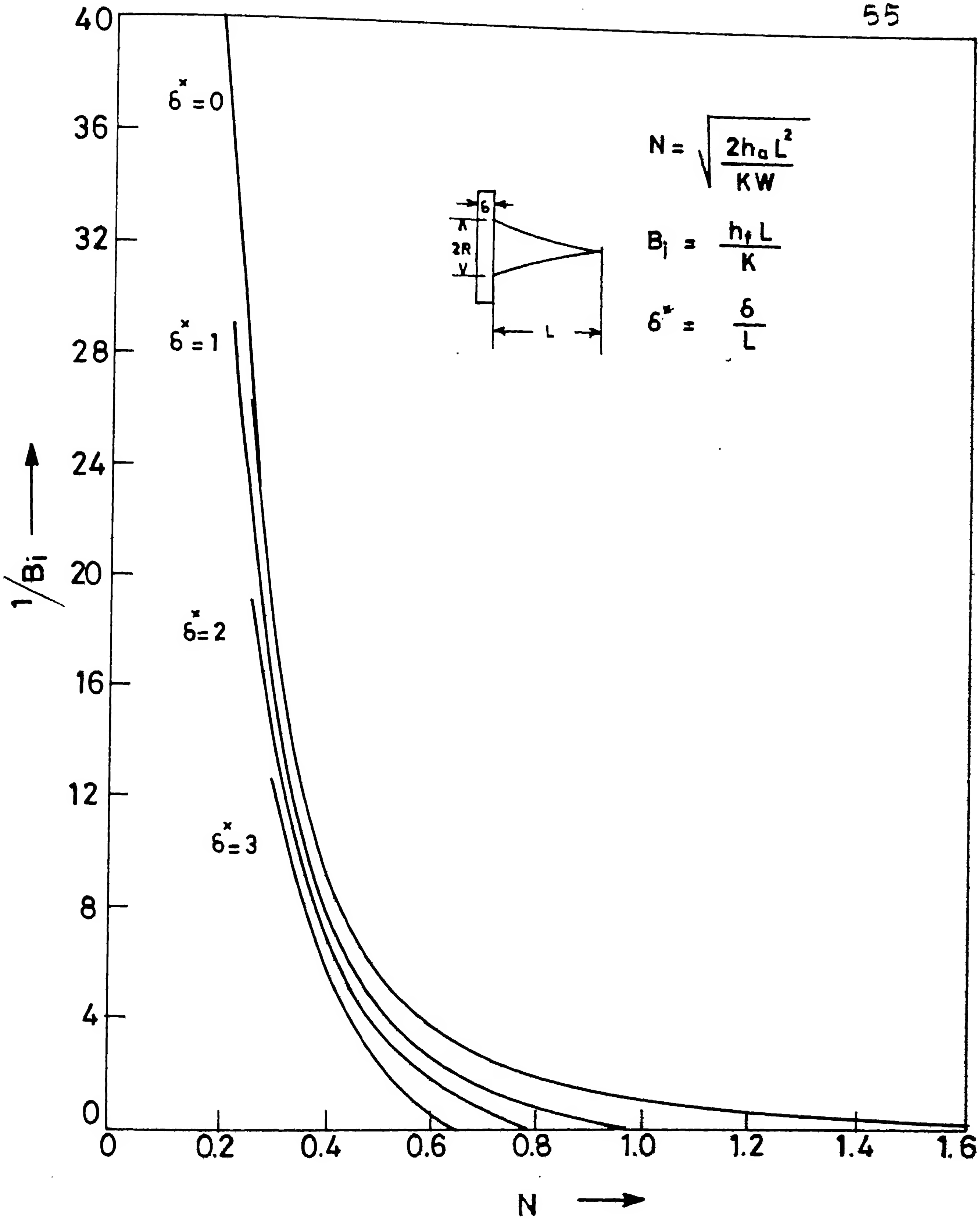

FIG.9 OPTIMUM DIMENSIONS FOR CONCAVE SPINE.

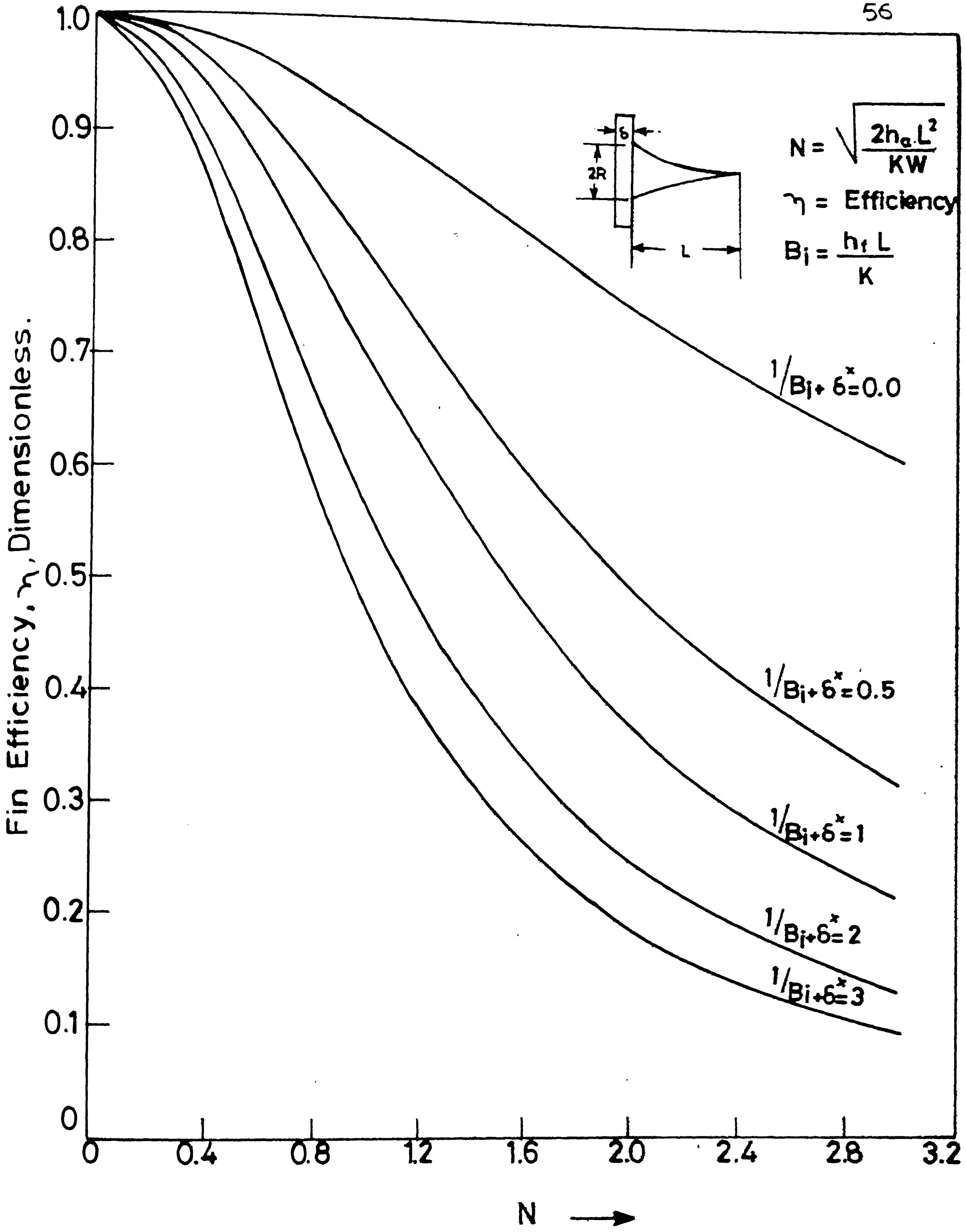

FIG. 10 FIN EFFICIENCY FOR CONCAVE SPINE.

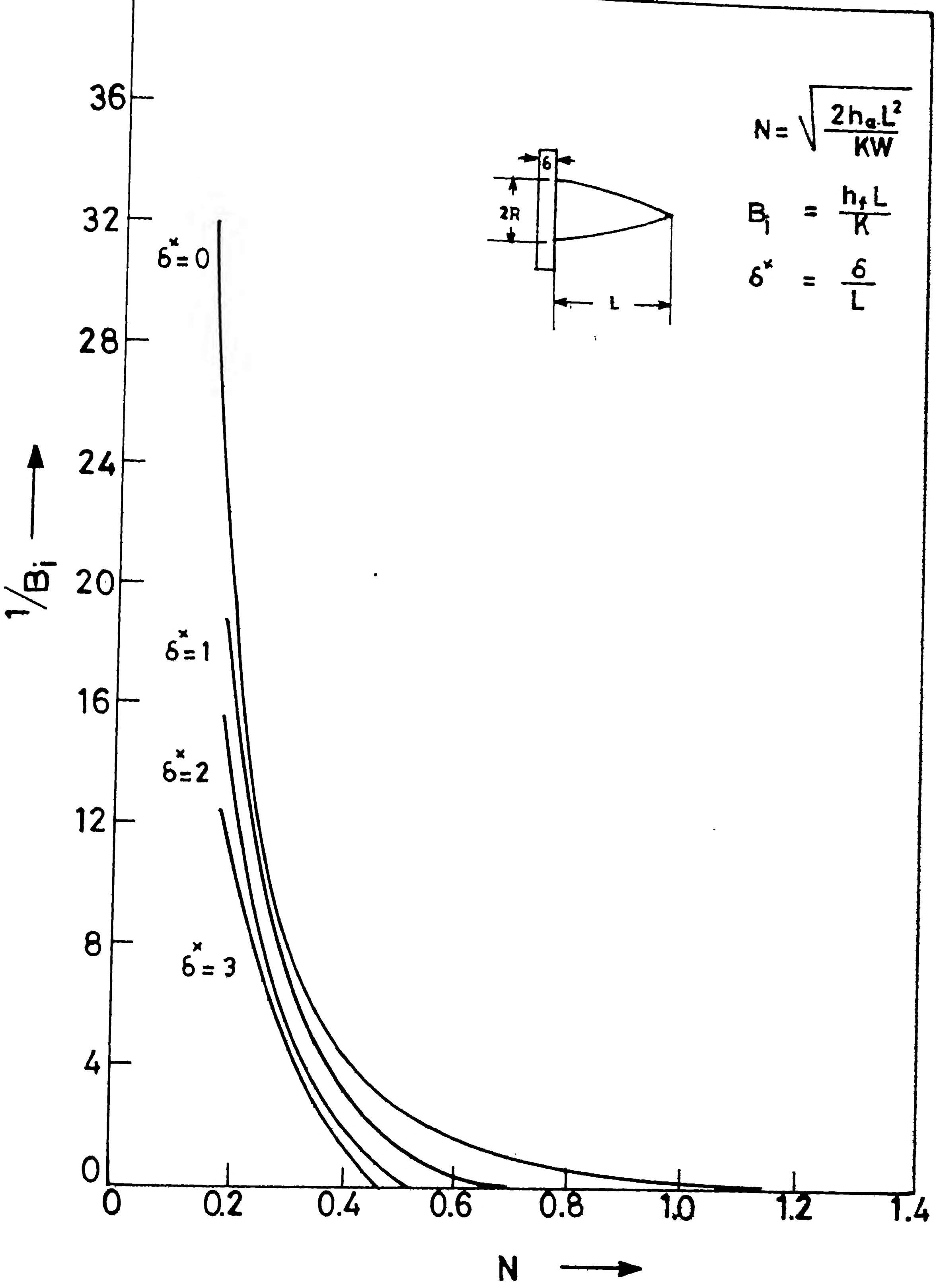

FIG. 11 OPTIMUM DIMENSIONS FOR CONVEX SPINE.

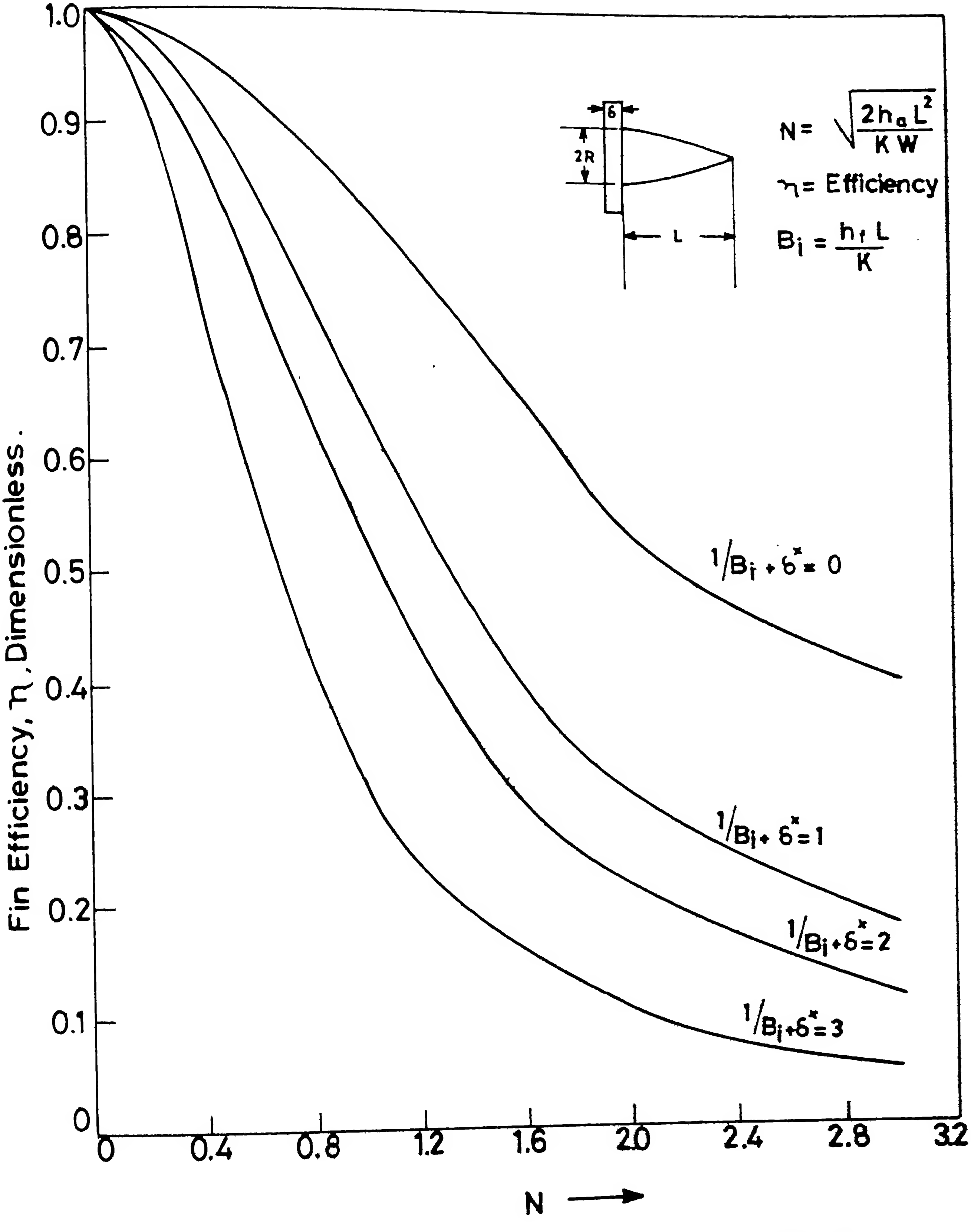

FIG. 12 FIN EFFICIENCY OF CONVEX SPINE.

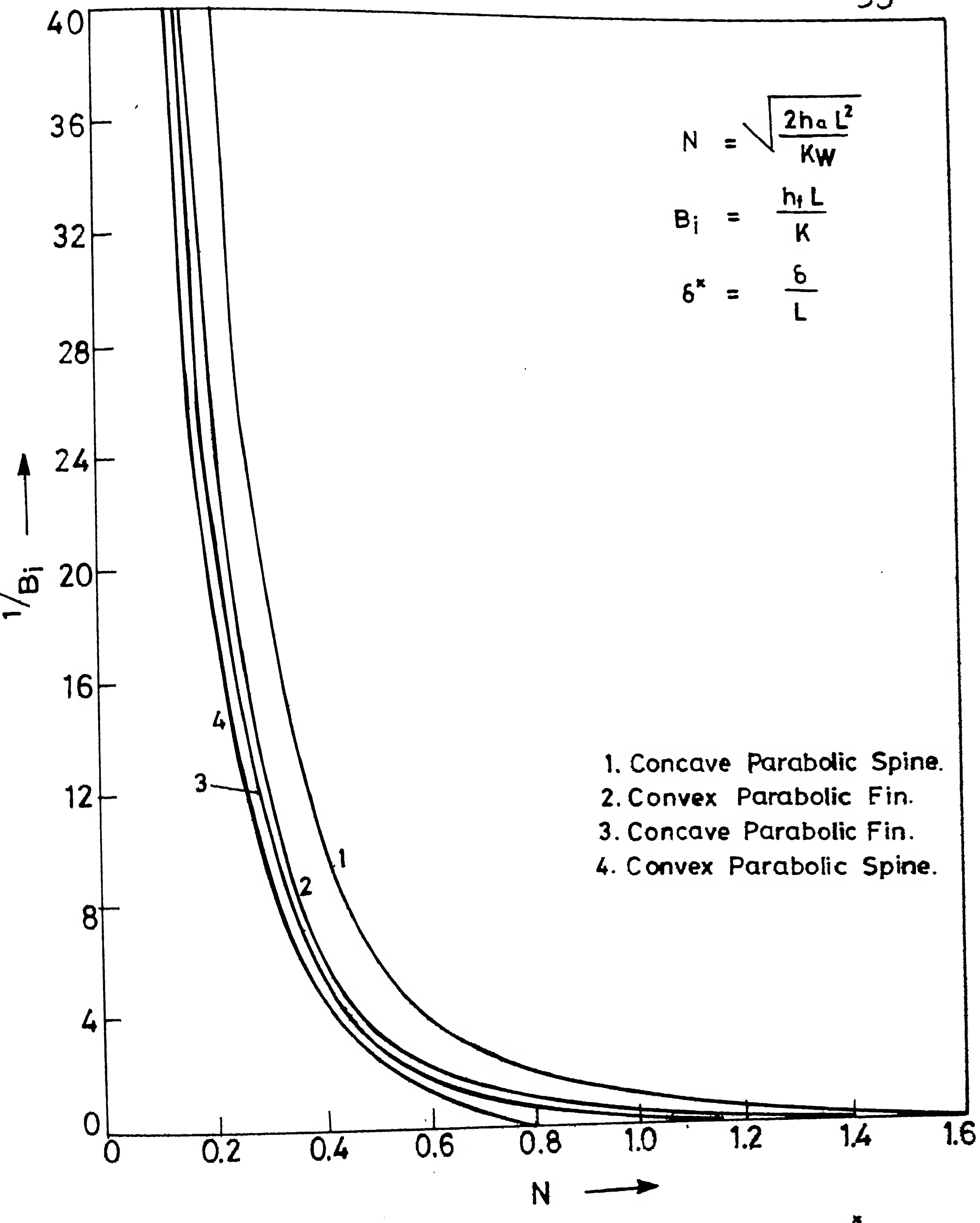

FIG. 13 OPTIMUM CURVES FOR VARIOUS FINS, $\delta^* = 0$.

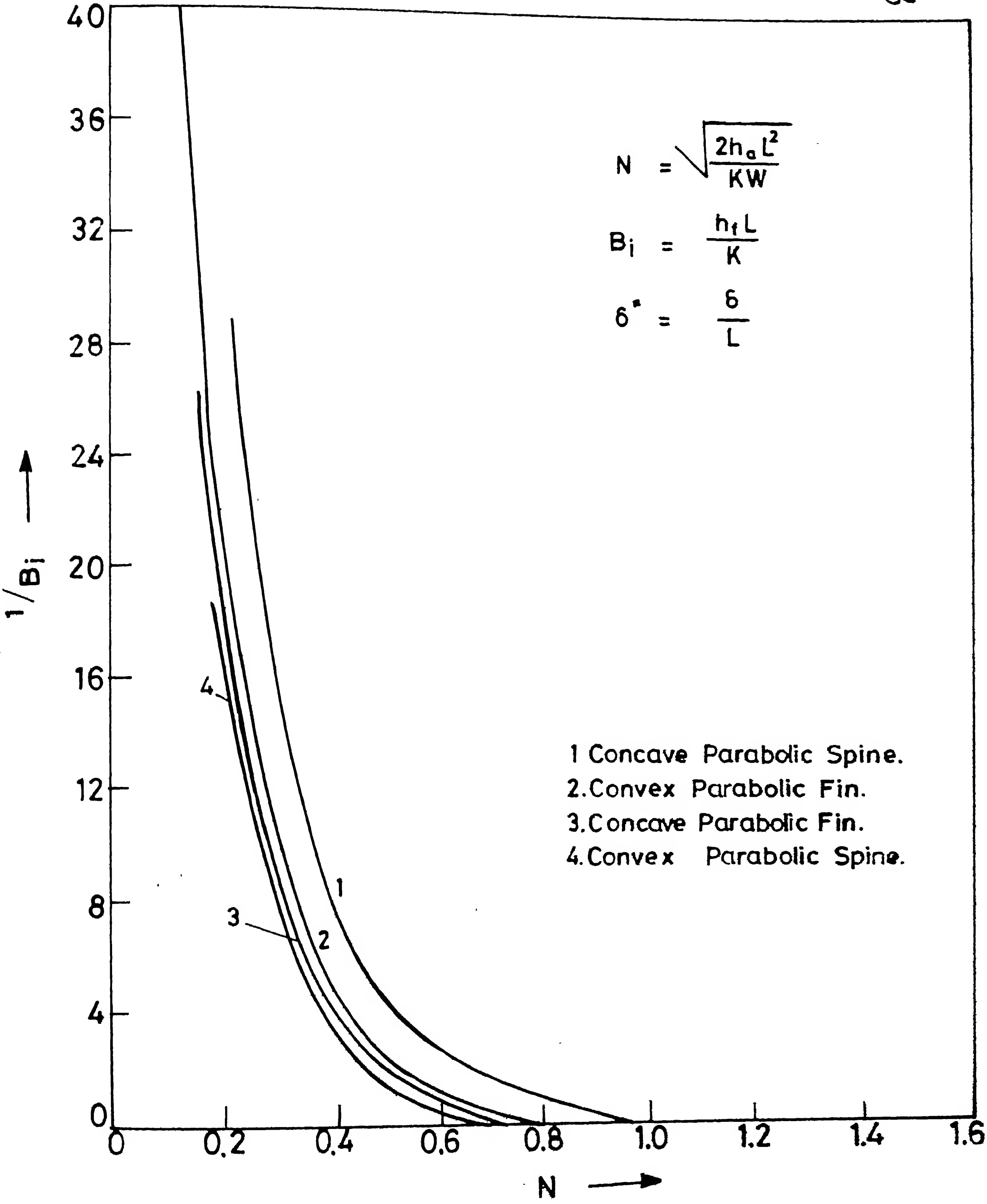

FIG.14 OPTIMUM CURVES FOR VARIOUS FINS, $\delta^*=1$.

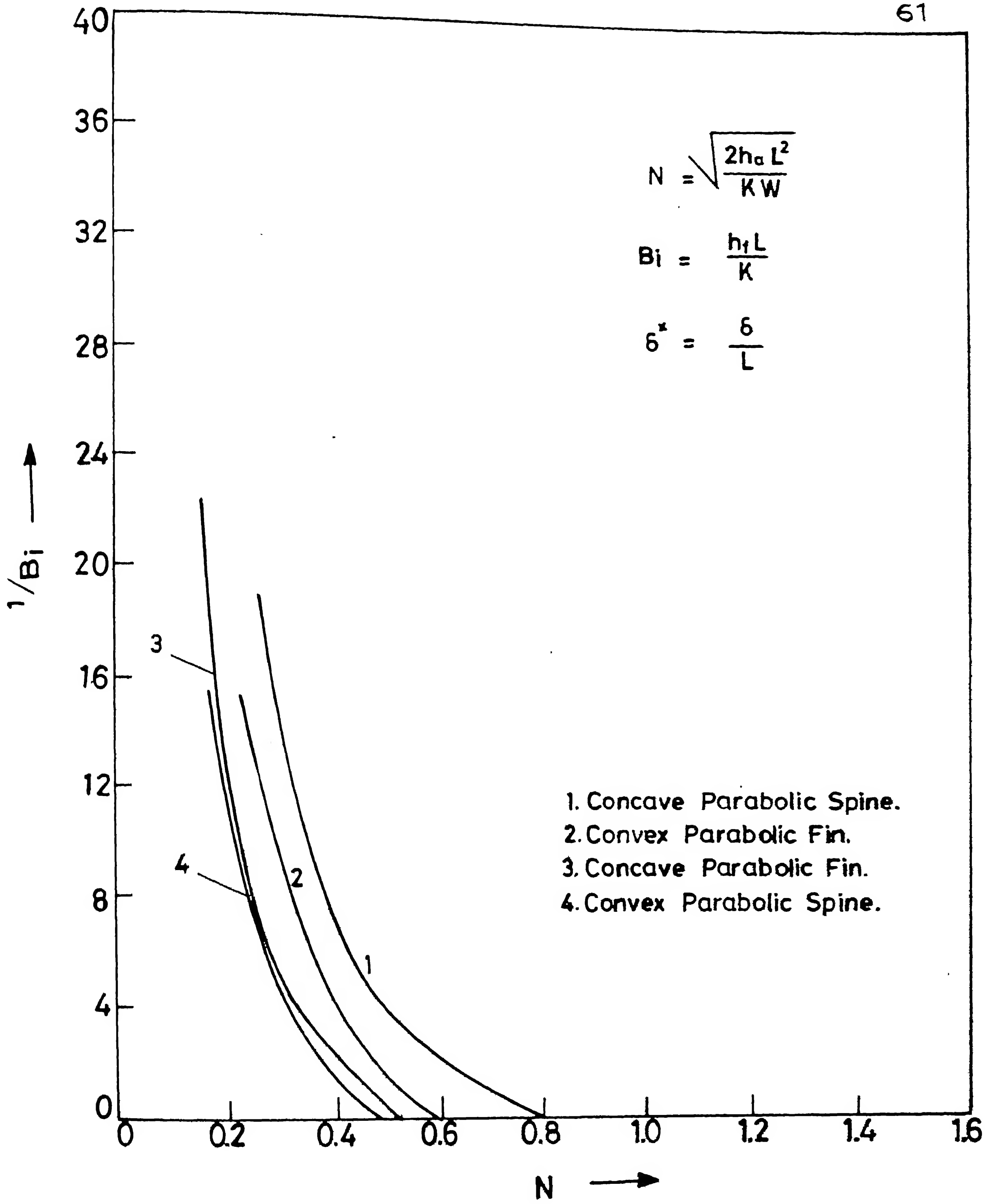

FIG. 15 OPTIMUM CURVES FOR VARIOUS FINS, $\delta^*=2$.

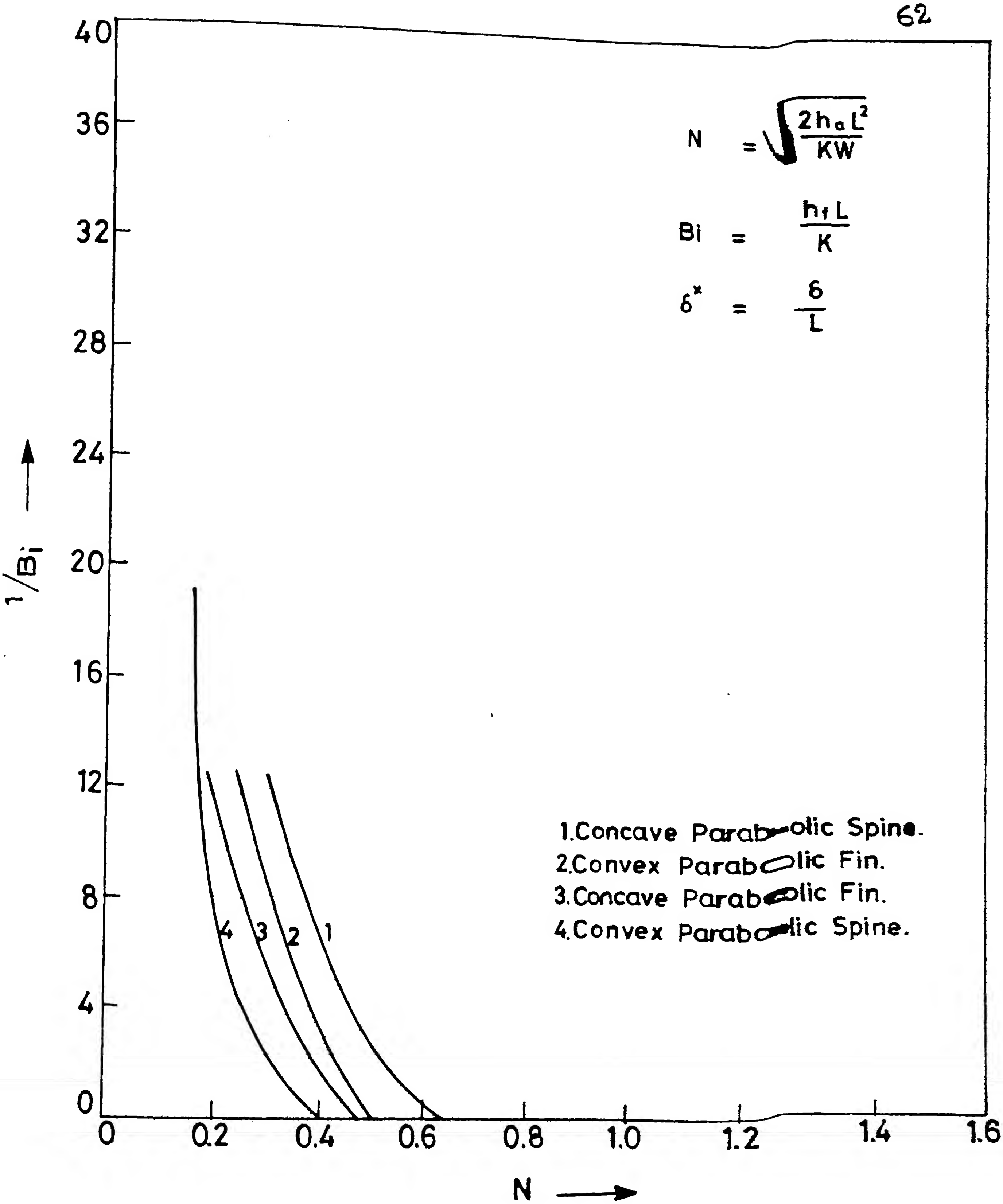

FIG. 16 OPTIMUUM CURVES FOR VARIOUS FINS, $\delta^*=3$.

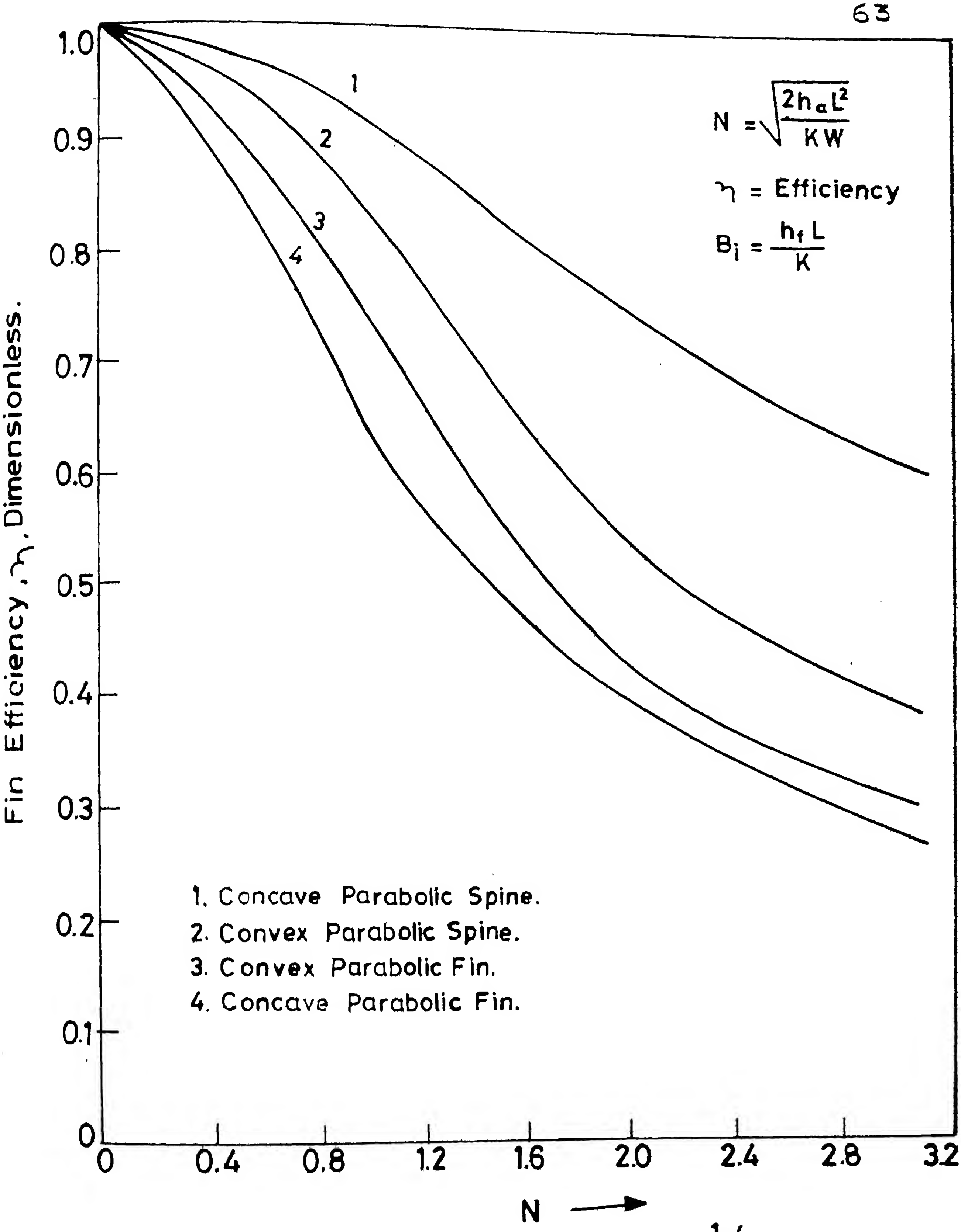

FIG. 17 EFFICIENCY OF VARIOUS FINS, $1/B_i + \delta^* = 0$.

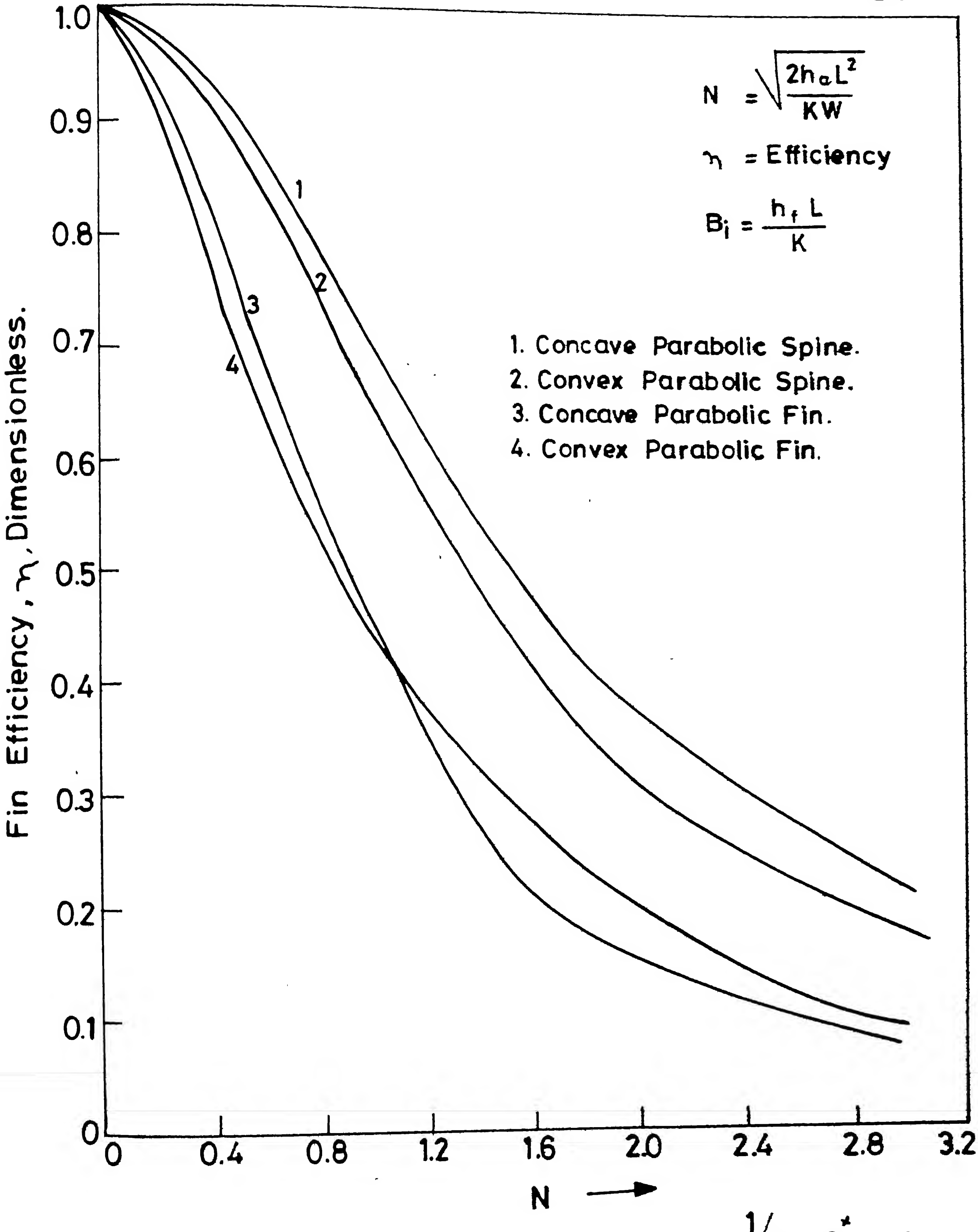

FIG. 18 EFFICIENCY OF VARIOUS FINS, $1/B_i + \delta^* = 1$

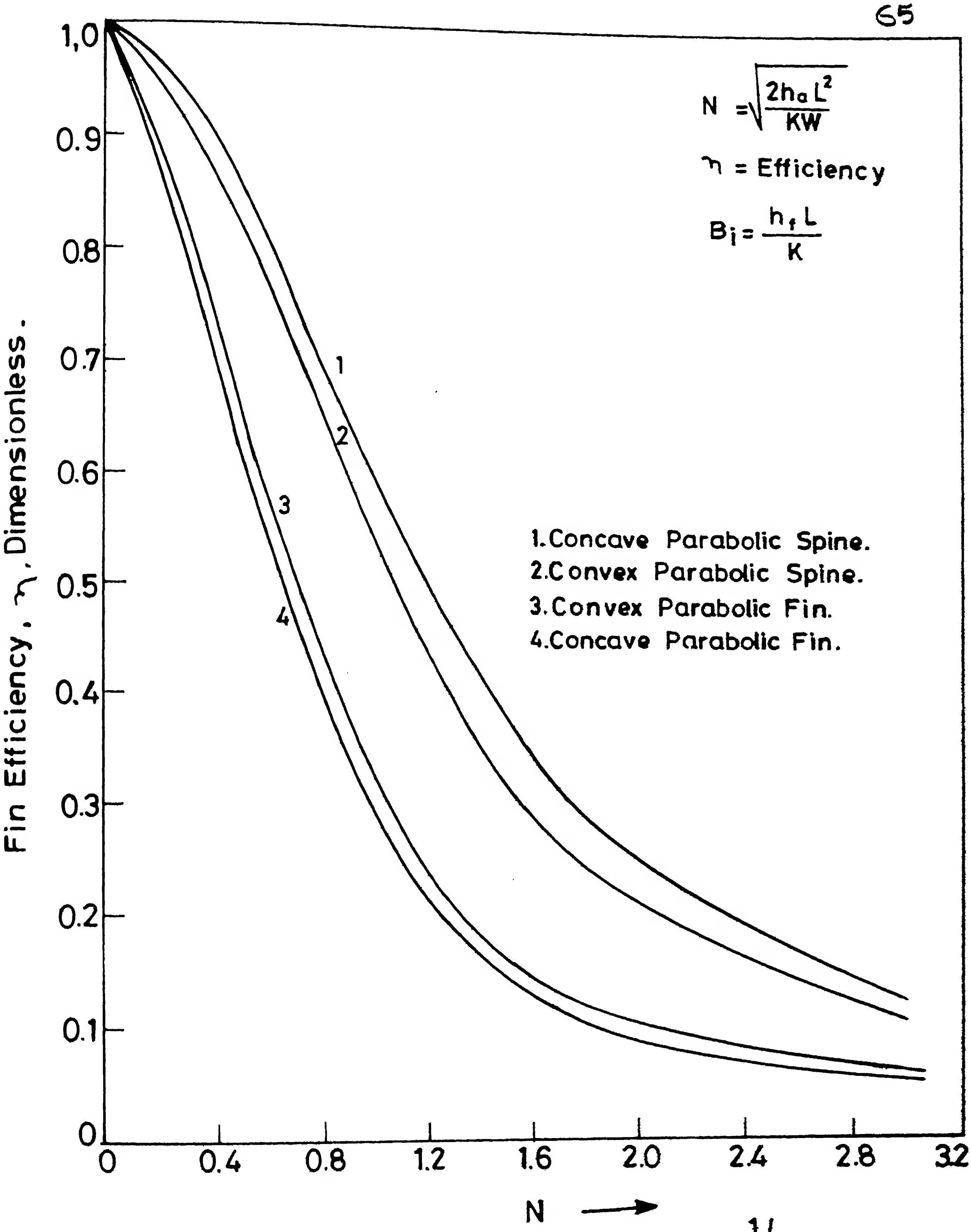

FIG.19 EFFICIENCY OF VARIOUS FINS, $1/B_i + \delta^* = 2$

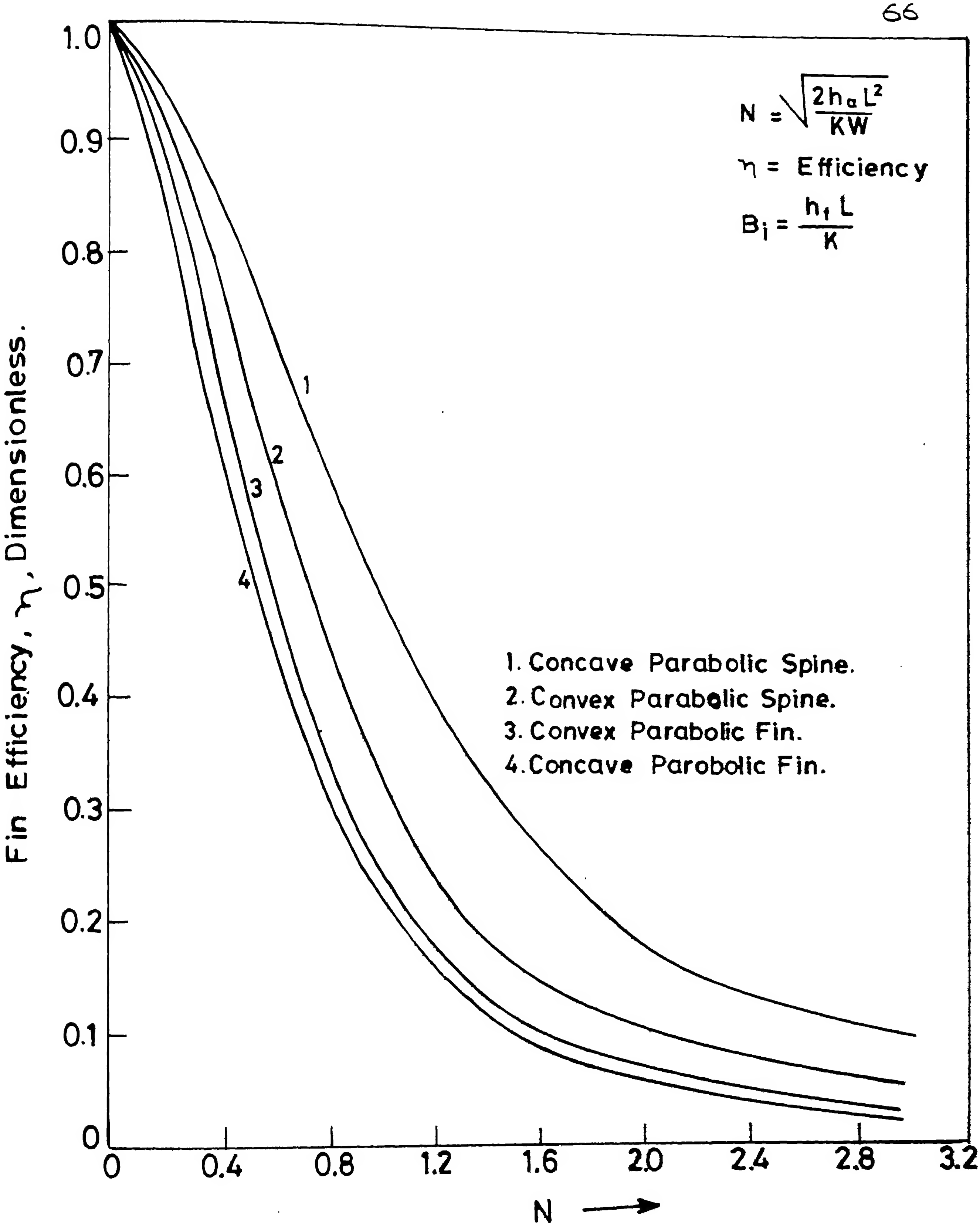

FIG. 20 EFFICIENCY OF VARIOUS FINS, $1/B_i + \delta^* = 3$

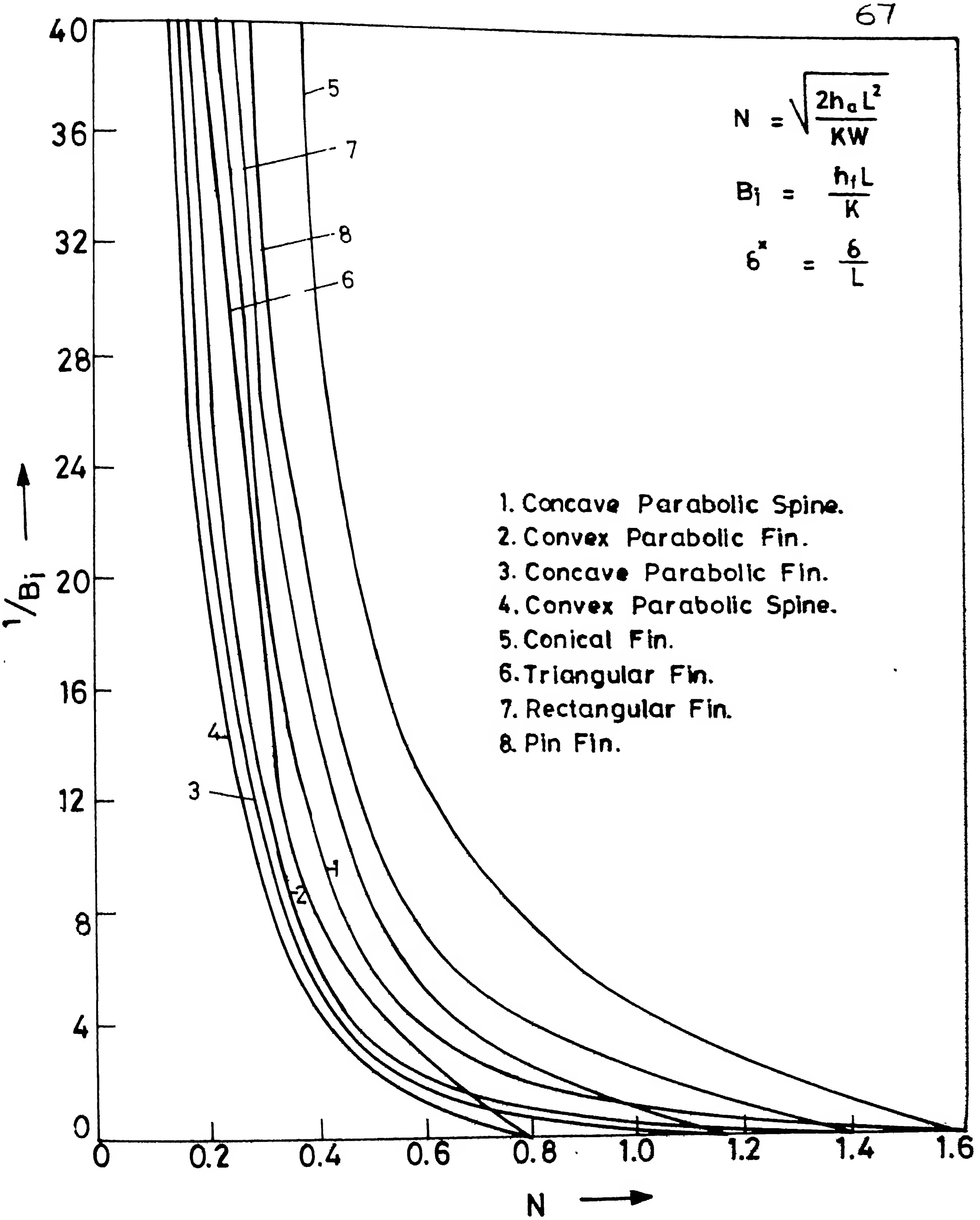

FIG. 21 COMPARISON OF OPTIMUM CURVES WITH OTHER PROFILES

$(\delta^* \simeq 0)$

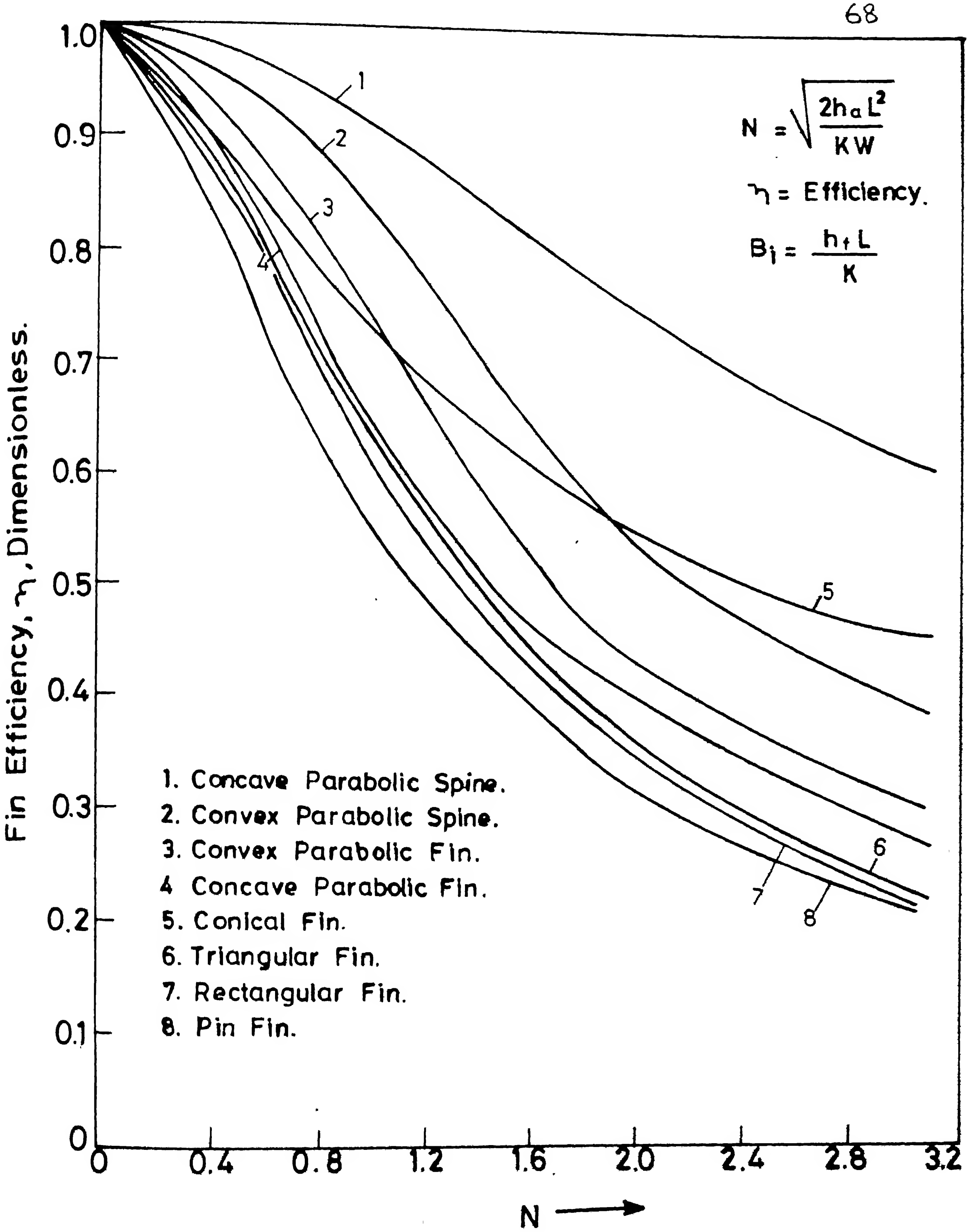

FIG.22 COMPARISON OF EFFICIENCY WITH OTHER PROFILES.

$(1/B_i + \delta^* = 0)$

PROGRAM LISTING

```
C      PROGRAMME FOR OBTAINING THE OPTIMAL DIMENSIONS OF FINS
       REAL K,L,N
       INTEGER FINNOM
       COMMON/BESSEL/BFM(600),BSM(600),BFN(600),BFP(600)
       OPEN(UNIT=1,FILE='OPTDIM.OUT')
       OPEN(UNIT=2,FILE='BFMBSM.INP')
       OPEN(UNIT=3,FILE='BFNBFP.INP')
       OPEN(UNIT=4,FILE='VALUES.INP')
       WRITE(*,3)
3      FORMAT(//////////5X,'PLEASE INPUT ONE OF THE FOLLOING NUMBERS TO
     1 SPECIFY'/5X,'THE TYPE OF FIN'                //
     1 10X,'1 = CONCAVE PARABOLIC FIN'//
     1 10X,'2 = CONVEX  PARABOLIC FIN'//
     1 10X,'3 = CONCAVE PARABOLIC SPINE'//
     1 10X,'4 = CONVEX  PARABOLIC SPINE'///
     1 40X,'THEN PRESS RETURN TO CONTINUE.....')
       READ(*,*)FINNOM
       READ(2,*)(BFM(I),I=1,599)
       READ(2,*)(BSM(I),I=1,599)
       READ(3,*)(BFN(I),I=1,600)
       READ(3,*)(BFP(I),I=1,600)
       READ(4,*)K,HF,HA,TF,TA,DELTA,QGIVEN
       PI=3.1416
       L=0.0
13     L=L+0.001
       B=K/(HF*L)
       N=FNDN(B,FINNOM)
       M=400*N
       W=(2.*HA*L**2)/(K*N**2)
       X=(K*TA)/L
       THETAF=(TF/TA-1.)
       GO TO (12,22,32,42),FINNOM
12     P=-0.5+0.5*SQRT(1.+4*N**2)
       Q=(X*P*THETAF*W)/(1.+P*(B+DELTA/L))
       ETA=P/((N**2)*(1.0+P*(B+DELTA/L)))
       WRITE(*,*)Q
       GO TO 52
22     Q=(X*THETAF*N*W*BFP(M))/(BFN(M)+N*BFP(M)*(B+DELTA/L))
       ETA=BFP(M)/(N*(BFN(M)+N*BFP(M)*(B+DELTA/L)))
       WRITE(*,*)Q
       GO TO 52
32     P=-1.5+0.5*SQRT(9.+4*N**2)
       Q=(X*P*THETAF*PI*W**2)/(1.+P*(B+DELTA/L))
       ETA=(P*3.)/((N**2)*(1.0+P*(B+DELTA/L)))
       WRITE(*,*)Q
       GO TO 52
42     Q=(X*THETAF*BSM(M)*N*PI*W**2)/(BFM(M)+N*BSM(M)*(B+DELTA/L))
       ETA=((1.5/N)*(BSM(M)))/((BFM(M)+N*BSM(M)*(B*DELTA/L)))
       WRITE(*,*)Q
       GO TO 52
52     IF(Q.LT.QGIVEN) GO TO 13
       IF(FINNOM.LE.2) WRITE(1,5)L,W,N
       IF(FINNOM.GT.2) WRITE(1,7)L,W,N
       IF(FINNOM.LE.2) WRITE(1,9)Q,ETA
```

```
      IF(FINNOM.GT.2) WRITE(1,11)Q,ETA
5     FORMAT(8X,'LENGTH OF THE FIN ='F7.3,'m'//8X,'WIDTH OF THE FIN
     1 = 'F7.3,'m'//8X,'DIMENSIONLESS PARAMETER, N = 'F7.4)
7     FORMAT(8X,'LENGTH OF THE SPINE ='F7.3,'m'//8X,'RADIUS OF THE
     1 SPINE ='F7.3,'m'//8X,'DIMENSIONLESS PARAMETER, N = 'F7.4)
9     FORMAT(/8X,'HEAT DISSIPATION OF THE FIN = 'F10.4,' WATTS/METRE'/
     1 /8X,'FIN EFFICIENCY = 'F9.6)
11    FORMAT(/8X,'HEAT DISSIPATION OF THE SPINE ='F8.4' WATTS'/
     1 /8X,'SPINE EFFICIENCY = 'F9.6)
      STOP
      END
C     FUNCTION FOR CALCULATING THE VALUE OF 'N'.
      FUNCTION FNDN(B,FINNOM)
      INTEGER FINNOM
      REAL  N1,N2
      COMMON/BESSEL/BFM(600),BSM(600),BFN(600),BFP(600)
      GO TO (12,22,12,32),FINNOM
12    DO 10 I=1,600
      BINV=FNDBIN(0.0025*I,FINNOM)
      IF (BINV.LE.B)GO TO 42
10    CONTINUE
42    IF(BINV.NE.B)GO TO 52
      FNDN=0.0025*I
      RETURN
52    N1=.0025*(I-1.)
      N2=.0025*I
13    FNDN=(N1+N2)/2.0
      IF (FNDBIN(FNDN,FINNOM).GE.B) GO TO 62
      N2=FNDN
      IF ((N2-N1).GT.1.0E-04) RETURN
      GO TO 13
62    N1=FNDN
      IF ((N2-N1).GT.1.0E-04)GO TO 13
      RETURN
22    DO 20 I=1,600
      BINV=(2.*(BFN(I)**2-BFP(I)**2)-(BFP(I)*BFN(I)/(I*.0025))
     1 /(BFP(1)**2)
      IF (BINV.LE.B) GO TO 72
20    CONTINUE
72    FNDN=0.0025*I
      RETURN
32    DO 30 I=1,599
      BINV=((5./3.)*(BFM(I)**2/BSM(I)**2-1.0))-((2.*BFM(I))
     1 /(BSM(I)*(.0025*I+.0025)))
      IF (BINV.LE.B)GO TO 82
30    CONTINUE
82    FNDN=0.0025*I
      RETURN
      END
C     FUNCTION FOR CALCULATING 1/Bi
      FUNCTION FNDBIN(N,FINNOM)
      INTEGER FINNOM
      REAL N
      GO TO (12,22,32,42),FINNOM
12    P=SQRT(1.0+4*N**2)
```

```
    1  /(2*P**2)
       RETURN
42     RETURN
       END
```